# Charge relaxation resistance in the Coulomb blockade problem

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(Received 16 May 2009; published 30 July 2009)

We study the dissipation in a system consisting of a small metallic island coupled to a gate electrode and to a massive reservoir via single tunneling junction. The dissipation of energy is caused by a slowly oscillating gate voltage. We compute it in the regimes of weak and strong Coulomb blockade. We focus on the regime of not very low temperatures when electron coherence can be neglected but quantum fluctuations of charge are strong due to Coulomb interaction. The answers assume a particularly transparent form while expressed in terms of specially chosen physical observables. We discovered that the dissipation rate is given by a universal expression in both limiting cases.

DOI: 10.1103/PhysRevB.80.035332 PACS number(s): 73.23.Hk, 73.43.Nq

## I. INTRODUCTION

The phenomenon of Coulomb blockade has become an excellent tool for observation of interaction effects in single-electron devices. Theoretical means for its exploration are well developed and versatile.[1–6] The simplest mesoscopic system displaying Coulomb blockade is a single-electron box (SEB). The properties of such a system are essentially affected by electron coherence and interaction. Our work is motivated by a considerable recent theoretical and experimental interest in the relation between dissipation and resistance of this device in various parametric regimes.[7–12]

The setup is as follows (see Fig. 1). Metallic island is coupled to an equilibrium electron reservoir via tunneling junction. The island is also coupled capacitively to the gate electrode. The potential of the island is controlled by the voltage $U_g$ of the gate electrode. The physics of the system is governed by several energy scales: the Thouless energy of an island $E_{\mathrm{Th}}$, the charging energy $E_c$, and the mean level spacing $\delta$. Throughout the paper the Thouless energy is considered to be the largest scale in the problem. This allows us to treat the metallic island as a zero-dimensional object with vanishing internal resistance. The dimensionless conductance of a tunneling junction $g$ is an additional control parameter.

Initially, the main quantity of interest in a Coulomb blockaded SEB was its effective capacitance $\partial Q/\partial U_g$, where $Q$ is the average charge of a island.[13–18] Paper[7] however sparked both theoretical and experimental attention to the dynamic response functions of such a setup.[8–12,19,20] It is worthwhile to mention that the system does not allow for conductance measurements since there is no dc transport. This way an essential dynamic characteristic becomes the setup admittance, which is a current response to an ac gate voltage $U_g(t)=U_0+U_\omega \cos \omega t$. As it is well known, the real part of admittance determines energy dissipation in an electric circuit. Classically, the average energy-dissipation rate of a single-electron box is given as follows:

$$\mathcal{W}_\omega = \omega^2 C_g^2 R |U_\omega|^2, \quad R = \frac{h}{e^2 g}, \quad \hbar\omega \ll g E_c, \tag{1}$$

where $C_g$ denotes the gate capacitance, $e$ the electron charge, and $h=2\pi\hbar$ the Planck constant. Expression (1) presents us with a natural way of extracting the resistance of a system from its dissipation power. The resistance of a classical system is thus fully determined by the tunneling conductance of the contact via Kirchhoff's law, $R=h/(e^2 g)$. The question one asks is how quantum effects such as electron coherence and interaction change this result? One expects that correct quantum dissipation is going to give generalized quantum resistance. The obvious stumbling block one foresees is that only combination of two observables: $C_g^2 R$ can be extracted from the dissipation power rather than just $R$. For the case of fully coherent SEB this key difficulty was resolved in Ref. 7. It was shown that the energy-dissipation rate $\mathcal{W}_\omega$ can be factorized in accordance with its classical appearance Eq. (1) but the definition of physical quantities comprising it becomes different. Geometrical capacitance $C_g$ should be substituted by a new observable mesoscopic capacitance $C_\mu$. This leads to the establishment of another observable *charge relaxation* resistance $R_q$ such that $R \to R_q$ in Eq. (1). Charge relaxation resistance of a coherent system differs drastically from its classical counterpart. In particular, as shown by Büttiker *et al.*,[7] the charge relaxation resistance of a single-channel junction does not depend on its transmission. The admittance in the quasistatic regime was investigated in the recent experiment by Gabelli *et al.*[11] The measurements were performed at low temperatures $T \lesssim \delta$ when the system could be regarded as coherent. The question that has remained unattended by the theory is what happens to dissipation and resistance at transient temperatures when thermal fluctuations smear out electron coherence but electron-electron interaction is strong? The recent experiment by Persson *et al.*[12]

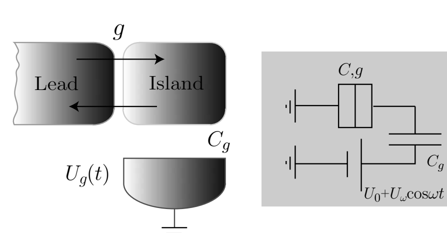


FIG. 1. Measurement of resistance $R_q$. The SEB is subjected to a constant gate voltage $U_0$ The dissipative current through the tunneling contact is caused by a weak ac voltage $U(t)$.

 

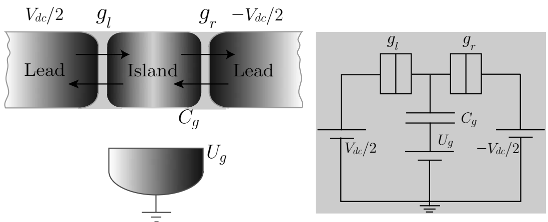


FIG. 2. Measurement of conductance. The SET is subjected to a constant gate voltage $U_g$ and constant bias $U$.

explored the energy-dissipation rate at these transient temperatures.

Motivated by the experiment[12] we address the same question from the theoretical point of view. We study the energy-dissipation rate of a single-electron box in the so-called "interactions without coherence" regime. It corresponds to the following hierarchy of energy scales: $E_{\mathrm{Th}} \gtrsim E_c \gg T \gg \max\{\delta, g\delta\}$. This temperature regime is such that keeps electrons strongly correlated ($T \ll E_c$), yet allows to discard electron coherence ($T \gg \max\{\delta, g\delta\}$).[21,22] We compute the energy-dissipation rate and the SEB admittance in the limits of large ($g \gg 1$) and small ($g \ll 1$) dimensionless tunneling conductance of the junction.

We consider a multichannel junction but the conductance of each channel is assumed to be small $g_{\mathrm{ch}} \ll 1$. Then, the physics of the system is most adequately described in the framework of Ambegaokar-Eckern-Schön (AES) effective action.[23] Our results lead to the generalization of classical result (1). We found that at $\omega \to 0$ the average energy-dissipation rate can be factorized in both $g \gg 1$ and $g \ll 1$ limits as

$$\mathcal{W}_\omega = \omega^2 C_g^2(T) R_q(T) |U_\omega|^2, \quad R_q(T) = \frac{h}{e^2 g'(T)}, \tag{2}$$

in complete analogy with classical expression (1). Here, $R_q(T)$ and $C_g(T)$ are identified as charge relaxation resistance and renormalized gate capacitance, respectively. It is worthwhile to mention that the physical observables $g'(T)$ and $C_g(T)$ are defined universally for any value of dimensionless conductance $g$. It allows us to suggest that Eq. (2) remains valid for arbitrary value of $g$.

In order to explain physics behind quantities $g'(T)$ and $C_g(T)$, it is useful to consider a single-electron transistor (SET) rather than SEB (see Fig. 2). In the absence of dc voltage between left and right reservoirs a SET represents essentially a SEB except different definition of the parameter $g$. Then, $g'(T)$ is the very quantity that determines the SET conductance. The renormalized gate capacitance $C_g(T)$ is very different from the effective capacitance $\partial Q / \partial U_0$. In fact, $C_g(T) = \partial q'(T) / \partial U_0$, where $q'(T)$ is the physical observable introduced recently in Ref. 24 to describe the $\theta$-angle renormalization in the Coulomb blockade problem. The quantity $q'$ is determined not only by the average charge $Q$ but also by the antisymmetrized (so-called, quantum) current noise in a SET.

The paper is organized as follows. Section II is used to introduce AES model. Sections III and IV are devoted to dissipation in the weak- ($g \gg 1$) and strong-coupling ($g \ll 1$) regimes. Section V is devoted to discussion.

## II. FORMALISM

### A. Hamiltonian

A single-electron box is described by the Hamiltonian

$$H = H_0 + H_c + H_t, \tag{3}$$

where $H_0$ describes free electrons in the lead and the island, $H_c$ describes Coulomb interaction of carriers in the island, and $H_t$ describes the tunneling,

$$H_0 = \sum_k \varepsilon_k^{(a)} a_k^\dagger a_k + \sum_\alpha \varepsilon_\alpha^{(d)} d_\alpha^\dagger d_\alpha. \tag{4}$$

Here, operators $a_k^\dagger$ ($d_\alpha^\dagger$) create a carrier in the lead (island),

$$H_t = \sum_{k,\alpha} t_{k\alpha} a_k^\dagger d_\alpha + \text{H.c.} \tag{5}$$

The charging Hamiltonian of electrons in the box is taken in the capacitive form

$$H_c = E_c(\hat{n}_d - q)^2. \tag{6}$$

Here, $E_c = e^2/(2C)$ denotes the charging energy and $q = C_g U_g / e$ the gate charge. $\hat{n}_d$ is an operator of a particle number in the island,

$$\hat{n}_d = \sum_\alpha d_\alpha^\dagger d_\alpha. \tag{7}$$

It is convenient to introduce Hermitian matrices

$$\hat{g}_{kk'} = (2\pi)^2 [\delta(\varepsilon_k^{(a)}) \delta(\varepsilon_{k'}^{(a)})]^{1/2} \sum_\alpha t_{k\alpha} \delta(\varepsilon_\alpha^{(d)}) t_{\alpha k'}^\dagger, \tag{8}$$

$$\hat{\hat{g}}_{\alpha\alpha'} = (2\pi)^2 [\delta(\varepsilon_\alpha^{(d)}) \delta(\varepsilon_{\alpha'}^{(d)})]^{1/2} \sum_k t_{\alpha k}^\dagger \delta(\varepsilon_k^{(a)}) t_{k\alpha'}, \tag{9}$$

the first of them acting in the Hilbert space of the states of the lead and the second in the space of the islands states. The energies $\varepsilon^{(a)}, \varepsilon^{(d)}$ are accounted for with respect to the Fermi level and the delta functions should be smoothed on the scale $\delta E$ such that $\delta \ll \delta E \ll T$.

The eigenstates of $\hat{g}$ ($\hat{\hat{g}}$) describe the "channel states" in the lead (island) while the transmittances of the corresponding channels $\mathcal{T}_\gamma$ are related to the eigenvalues $g_\gamma$. Note that, in general, the rank of the matrix $\hat{g}$ differs from that of the matrix $\hat{\hat{g}}$ so that the numbers of eigenvalues are also different. This difference is, however, irrelevant since it stems from the "closed channels" with $g_\gamma \approx 0$, i.e., the states strongly localized either within the lead or within the island. The effective "channel conductance" $g_{\mathrm{ch}}$ and the effective number of open channels $N_{\mathrm{ch}}$ can be defined as[25]

$$g_{\mathrm{ch}} = \frac{\mathrm{tr}(\hat{g}^2)}{\mathrm{tr}\,\hat{g}}, \quad N_{\mathrm{ch}} = \frac{(\mathrm{tr}\,\hat{g})^2}{\mathrm{tr}(\hat{g}^2)}. \tag{10}$$

In general case the effective action can be written as a sum of terms, proportional to $\mathrm{tr}(\hat{g}^k)$, over all integer $k$ (see Ref. 25).

The problem is considerably simplified in the tunnel case, when

$$g_{\rm ch} \ll 1. \tag{11}$$

In the present paper we assume this condition to be satisfied. Then all terms with $k>1$ can be neglected, compared to the leading term with $k=1$ and the standard form of the AES action can be easily reproduced. In particular, the classical dimensionless conductance of the junction is expressed as

$$g = \mathrm{tr}\, \hat{g} = \mathrm{tr}\, \hat{\hat{g}} = g_{\rm ch} N_{\rm ch}. \tag{12}$$

Note, that under the condition (11) $g$ still can be large, if the number of channels $N_{\rm ch} \gg 1$ is sufficiently large.

Throughout the paper we keep the units such that $\hbar = e = 1$, except for the final results.

### B. Conductance and dissipation

To study the electric properties of a system we compute energy dissipation caused by slow oscillations of external gate voltage $U_g(t) = U_0 + U_\omega \cos \omega t$.

The average energy-dissipation rate can be found by following the standard scheme:[26]

$$\mathcal{W}_\omega = \frac{dE}{dt} = \left\langle \frac{\delta H}{\delta U_g} \right\rangle \frac{dU_g}{dt}. \tag{13}$$

Here, $E$ is the energy of the system, $H$ is given by Eq. (3) and angular brackets denote full quantum statistical average. Since

$$\left\langle \frac{\delta H}{\delta U_g} \right\rangle = -\frac{C_g}{C} \sum_\alpha \langle d_\alpha^\dagger d_\alpha \rangle + \frac{C_g^2}{C} U_g, \tag{14}$$

the energy dissipation is determined by a response of the electron density in the island to the time-dependent gate voltage $U_g(t)$. Therefore, it can be found via Callen-Welton fluctuation-dissipation theorem[27]

$$\mathcal{W}_\omega = \frac{C_g^2}{2C^2} \omega \, \mathrm{Im}\, \Pi^R(\omega) |U_\omega|^2. \tag{15}$$

Here, $\Pi^R(\omega)$ is the retarded electron polarization operator,

$$\Pi^R(t) = i\Theta(t) \langle [\hat{n}_d(t), \hat{n}_d(0)] \rangle, \quad \hat{n}_d = \sum_\alpha d_\alpha^\dagger d_\alpha \tag{16}$$

with $\Theta(t)$ denoting Heaviside step function.

We are interested in the quasistatic regime $\omega \to 0$. Then, as it will be proven below, the polarization operator $\Pi^R(\omega)$ is possible to expand in regular series in $\omega$,

$$\Pi^R(\omega) = \pi_0(T) + i\omega \pi_1(T) + \mathcal{O}(\omega^2), \tag{17}$$

where both $\pi_0(T)$ and $\pi_1(T)$ are real functions of temperature and other SEB parameters. Then the energy-dissipation rate is solely determined by the linear coefficient $\pi_1(T)$ and acquires Ohmic form

$$\mathcal{W}_\omega = \frac{\omega^2}{2} \mathcal{A}(T) |U_\omega|^2, \quad \mathcal{A}(T) = \frac{C_g^2}{C^2} \pi_1(T). \tag{18}$$

The SEB admittance $g(\omega)$ which is the linear reponse of an ac current $I_\omega$ to ac gate voltage $U_\omega$: $\mathcal{G}(\omega) = I_\omega / U_\omega$, is related to the polarization operator (see Appendix B)

$$\mathcal{G}(\omega) = -i\omega C_g [1 + \Pi^R(\omega)/C]. \tag{19}$$

As expected, the energy-dissipation rate is proportional to the real part of the admittance, $\mathcal{W}_\omega \sim \mathrm{Re}\, \mathcal{G}(\omega)$. The static part of the polarization operator $\Pi^R(\omega)$ is determined by the effective capacitance $\partial Q / \partial U_0$ as

$$\pi_0(T) = \frac{C}{C_g} \frac{\partial Q}{\partial U_0} - C, \tag{20}$$

where $Q = \langle \hat{n}_d \rangle$ denotes the average charge on the island. We mention that Eq. (20) is analogous to the well-known Ward identity which relates static polarization operator and compressibility.[28] Using Eqs. (17)–(20), we can establish the following result:

$$\mathcal{G}(\omega) = -i\omega \frac{\partial Q}{\partial U_0} + \frac{C}{C_g} \mathcal{A}(T) \omega^2 + \mathcal{O}(\omega^3) \tag{21}$$

which is a quantum generalization of the classical relation

$$\mathcal{G}(\omega) = -i\omega C_g + C_g C R \omega^2 + \mathcal{O}(\omega^3). \tag{22}$$

Therefore, both the admittance and the energy-dissipation rate are determined by the polarization operator $\Pi^R(\omega)$ which involves one unknown function $\pi_1(T)$ in the quasistatic regime.

### C. AES model

The condition (11) validates the use of AES effective action[23] which describes the physics of the setup in terms of a single-quantum phase $\varphi(\tau)$ fluctuating in Matsubara time $\tau$

$$S_{\rm AES} = S_d + S_g + S_c. \tag{23}$$

Here, $S_d$ is the dissipative part of the action in the standard form

$$S_d = -\frac{g}{4} \int_0^\beta \alpha(\tau_{12}) e^{i\varphi(\tau_1) - i\varphi(\tau_2)} d\tau_1 d\tau_2,$$

$$\alpha(\tau) = \frac{T^2}{\sin^2 \pi T \tau} = -\frac{T}{\pi} \sum_{\omega_n} |\omega_n| e^{-i\omega_n \tau}, \tag{24}$$

where $\beta = 1/T$, $\tau_{12} = \tau_1 - \tau_2$, $\omega_n = 2\pi T n$, and $g$ is defined by Eq. (12) and stands for the dimensionless (in units $e^2/h$) conductance of the tunnel junction. The term $S_g$ represents a coupling with the gate voltage $U_0$,

$$S_g = -iq \int_0^\beta \dot{\varphi} d\tau = -2\pi q W i. \tag{25}$$

Here, integer $W$ is the winding number of a field $\varphi(\tau)$ which appears through the constraint

$$\varphi(\beta) - \varphi(0) = 2\pi W. \tag{26}$$

Nonzero value of $S_g$ appears for topologically nontrivial field configurations only. The charging part of the action is as follows:

$$S_c = \frac{1}{4E_c}\int_0^\beta \dot{\varphi}^2 d\tau. \qquad (27)$$

Physically, time derivative of the phase variable $\dot{\varphi}$ describes voltage fluctuations in a SEB. We emphasize that AES action is valid for any value of $g$. We work in the regime $T \ll E_c$. Charging term $S_c$ is thus always small providing a natural ultraviolet cutoff in the theory, $\Lambda = gE_c$.

Our aim is to compute the polarization operator Eq. (16) which, according to Eqs. (18) and (19), determines the energy dissipation and admittance. Therefore, we need to express initial observables cast in terms of fermionic operators through correlators of bosonic field $\varphi(\tau)$. This is done in Appendix A by employing Keldysh formalism. The polarization operator $\Pi^R(\omega)$ then can be obtained by analytical continuation $i\omega_n \to \omega + i0$ of the following phase correlator in Matsubara basis:

$$\Pi(\tau) = -C^2\langle \mathcal{T}_\tau \dot{\varphi}(\tau)\dot{\varphi}(0)\rangle. \qquad (28)$$

Here $\mathcal{T}_\tau$ denotes time ordering. So far we made no assumptions about the value of $g$. The AES model is however impossible to tackle for arbitrary $g$s due to highly nonlinear form of the dissipative term. In the next chapter of the paper we restrict our attention to the case of large dimensionless conductance $g \gg 1$; the quantity $1/g$ then becomes an expansion parameter of perturbation theory.

## III. WEAK-COUPLING REGIME, $g \gg 1$

### A. Perturbation theory

To expand the polarization operator $\Pi(i\omega_n)$ in powers of $1/g$ it is convenient to use the Matsubara frequency representation,

$$\varphi(\tau) = \sum_n \varphi_n e^{-i\omega_n \tau}, \quad \varphi_{-n} = \varphi_n^*. \qquad (29)$$

Then, the quadratic part of AES action assumes the form

$$S_{\text{AES}}^{(2)} = g\sum_{n>0}\left(n + \frac{2\pi^2 T}{gE_c}n^2\right)|\varphi_n|^2. \qquad (30)$$

It determines the propagator of the $\varphi$ field as

$$\langle \varphi_n \varphi_m\rangle = \frac{1}{g}\frac{\delta_{m,-n}}{|n| + 2\pi^2 T n^2/(gE_c)}. \qquad (31)$$

Evaluation of the polarization operator at the tree level yields

$$\frac{\Pi(i\omega_n)}{C^2} = -\frac{2\pi|\omega_n|}{g} + \mathcal{O}(\omega_n^2). \qquad (32)$$

Performing standard one-loop calculations one finds

$$\frac{\Pi(i\omega_n)}{C^2} = -\frac{2\pi|\omega_n|}{g}\left(1 + \frac{2}{g}\ln\frac{gE_c e^{\gamma+1}}{2\pi^2 T}\right) + \mathcal{O}(\omega_n^2). \qquad (33)$$

With the help of the renormalization-group analysis this result can be written as[29]

$$\frac{\Pi(i\omega_n)}{C^2} = -\frac{2\pi|\omega_n|}{g(T)} + \mathcal{O}(\omega_n^2). \qquad (34)$$

Here, $g(T)$ is given by

$$g(T) = g - 2\ln\frac{gE_c e^{\gamma+1}}{2\pi^2 T} \qquad (35)$$

with $\gamma \approx 0.577$ being Euler's constant. Equation (35) describes the well-known one-loop temperature renormalization of the coupling constant.[30]

### B. Instantons

So far the phenomenon of Coulomb blockade, i.e., dependence on $q$, is completely absent in all our expressions for polarization operator. To catch it we have to take into account instanton solutions of AES action.[31,32] Korshunov's instantons read

$$e^{i\varphi_W(\tau|\{z_a\})} = \prod_{a=1}^{|W|}\left[\frac{e^{2\pi i\tau T} - z_a}{1 - z_a^* e^{2\pi i \tau T}}\right]^{\operatorname{sgn} W}. \qquad (36)$$

Here, $z_a$ is a set of arbitrary complex numbers. Positive values of winding numbers $W$ are assigned to instantons with $|z_a| < 1$ and negative ones to anti-instantons with $|z_a| > 1$. On the classical solutions (36) the dissipative $S_d$ and topological $S_g$ part of AES action becomes

$$S_d[\varphi_W] + S_g[\varphi_W] = \frac{g}{2}|W| - 2\pi W q i. \qquad (37)$$

It is finite and independent of $z_a$s. These parameters are zero modes. The charging term though does depend on them,

$$S_c[\varphi_W] = \frac{\pi^2 T}{E_c}\sum_{a,b}\frac{1 + z_a z_b^*}{1 - z_a z_b^*}. \qquad (38)$$

Thus $z_a$s can only be viewed as approximate zero modes and the instanton configurations with $|z_a| \to 1$ are supressed.

As it is clear from Eq. (37) every instanton brings a small factor $e^{-g/2}$ to any observable we want to compute. In what follows, we restrict ourselves to one-instanton ($W = \pm 1$) contribution only.

### C. Instanton correction to the polarization operator

To get the instanton contribution to the polarization operator we need to compute one-instanton correction to the correlator $\langle \mathcal{T}_\tau \dot{\varphi}(\tau)\dot{\varphi}(0)\rangle$. Up to the one-instanton contributions we find

$$-\frac{\Pi(i\omega_n)}{C^2} \simeq \langle \dot{\varphi}\dot{\varphi}\rangle_{\omega_n}^{(0)}\left(1 - \sum_{W=\pm 1}\frac{\mathcal{Z}_W}{\mathcal{Z}_0}\right) + \sum_{W=\pm 1}\langle \dot{\varphi}\dot{\varphi}\rangle_{\omega_n}^{(W)} = \text{I} + \text{II}, \qquad (39)$$

where $\langle \dot{\varphi}\dot{\varphi}\rangle_{\omega_n} = \int_0^\beta \langle \dot{\varphi}(\tau)\dot{\varphi}(0)\rangle \exp(i\omega_n \tau) d\tau$ and

$$\mathcal{Z}_W = \int_W \mathcal{D}\varphi \exp[-S_{\text{AES}}],$$

$$\langle\dot{\varphi}\dot{\varphi}\rangle^{(W)} = \frac{1}{\mathcal{Z}_0}\int_W \mathcal{D}\varphi\dot{\varphi}(\tau)\dot{\varphi}(0)\exp[-S_{\rm AES}]. \quad (40)$$

Here, the subscript $W$ at the integral sign means that functional integration is performed over phase configurations obeying the boundary condition Eq. (26). The first term I in Eq. (39) represents the renormalization of the partition function due to instantons. The second term II is the contribution of the instanton solutions $\varphi_{\pm 1}$ into the correlation function itself. The renormalized partition function reads[16,18,25,33]

$$1 - \sum_{W=\pm 1}\frac{\mathcal{Z}_W}{\mathcal{Z}_0} = 1 - \frac{g^2 E_c}{\pi^2 T}e^{-g/2}\ln\frac{E_c}{T}\cos 2\pi q. \quad (41)$$

The contribution II consists of two terms

$$\mathrm{II} = \sum_{W=\pm 1}\langle\dot{\varphi}\dot{\varphi}\rangle^{(W)}_{\omega_n} = \sum_{W=\pm 1}\langle\dot{\varphi}_W\dot{\varphi}_W\rangle^{(W)}_{\omega_n} + \sum_{W=\pm 1}\langle\delta\dot{\varphi}_W\delta\dot{\varphi}_W\rangle^{(W)}_{\omega_n}, \quad (42)$$

where the first term is a correlator of classical field configurations Eq. (36) averaged over zero modes $z_a$ and the second term comes from fluctuations of phase $\varphi$ around the classical solution $\varphi_W$. As shown in Appendix B the latter term in Eq. (42) cancels the correction coming from the partition function Eq. (41). Therefore,

$$-\frac{\Pi(i\omega_n)}{C^2} = \langle\dot{\varphi}\dot{\varphi}\rangle^{(0)} + \sum_{W=\pm 1}\langle\dot{\varphi}_W\dot{\varphi}_W\rangle^{(W)}_{\omega_n}. \quad (43)$$

The first term in the rhs of Eq. (43) has been evaluated in Sec. III A. As it always happens in instanton physics,[34] the derivative $\dot{\varphi}_W(\tau)$ coincides with a zero mode of the fluctuation $\delta\varphi_W(\tau)$. It is worthwhile to mention that only zero modes of fluctuations around instanton solution contribute to the nonperturbative renormalization of the polarization operator. The corresponding contribution is as follows (see Appendix C for details):

$$\sum_{W=\pm 1}\langle\dot{\varphi}_W\dot{\varphi}_W\rangle^{(W)}_{\omega_n} = 4g^2E_c\left(\ln\frac{E_c}{T} - \frac{\pi|\omega_n|}{12T}\right)e^{-g/2}\cos 2\pi q + \mathcal{O}(\omega_n^2). \quad (44)$$

From Eqs. (34) and (44) we obtain

$$\frac{\Pi(i\omega_n)}{C^2} = -\frac{2g^2}{C}e^{-g/2}\ln\frac{E_c}{T}\cos 2\pi q - 2\pi|\omega_n| \times\left[\frac{1}{g(T)} - Dge^{-g(T)/2}\cos 2\pi q\right] + \mathcal{O}(\omega_n^2), \quad (45)$$

where constant $D = (\pi^2/3)\exp(-\gamma - 1)$.

The average charge on the island can be expressed via the partition function as

$$Q = q + \frac{T}{2E_c}\frac{\partial \ln \mathcal{Z}}{\partial q}. \quad (46)$$

Using Eq. (41) we find the following temperature and gate-voltage dependence of the average charge in the one-instanton approximation:

$$Q = q - \frac{g^2}{\pi}e^{-g/2}\ln\frac{E_c}{T}\sin 2\pi q. \quad (47)$$

Performing standard analytic continuation in Eq. (45), we obtain the retarded polarization operator $\Pi^R(\omega)$ in the form of Eq. (17) with $\pi_0(T)$ satisfying Eq. (20) and

$$\pi_1(T) = 2\pi C^2\left[\frac{1}{g(T)} - Dge^{-g(T)/2}\cos 2\pi q\right]. \quad (48)$$

Finally, the average energy-dissipation rate will be given by Eq. (18) with function

$$\mathcal{A}(T) = \frac{2\pi C_g^2}{g(T)}[1 - Dg^2(T)e^{-g(T)/2}\cos 2\pi q]. \quad (49)$$

In deriving this result we changed $g$ to $g(T)$ in the factor in front of the exponent in the rhs of Eq. (48). It is allowed by the accuracy we are working within. Result (49) asks for an interpretation. As expected, Coulomb blockade manifests itself as a periodic dependence of dissipation $A(T)$ on gate charge $q$. If we ascribe this dependence to the quantum resistance only, i.e., we write $\mathcal{A}(T) = C_g^2 R_q(T)$ with $R_q(T)$ following from Eq. (49) we face a paradox. It is believed that Coulomb blockade should suppress the tunneling of electrons between the island and the lead stronger for integer values of $q$ than for the half-integer ones. Therefore, it would be natural to expect that $R_q(T)$ is smaller at a half-integer value of $q$ than at an integer one. The discussion above suggests that we have to conceive some kind of temperature renormalization of the gate capacitance $C_g$.

### D. Physical observables and gate-capacitance renormalization

As shown in Ref. 24, the proper physical observables for the Coulomb blockade problem are

$$g'(T) = 4\pi\,\mathrm{Im}\left.\frac{\partial K^R(\omega)}{\partial\omega}\right|_{\omega=0},$$

$$q'(T) = Q + \mathrm{Re}\left.\frac{\partial K^R(\omega)}{\partial\omega}\right|_{\omega=0}, \quad (50)$$

where the average charge $Q$ is given by Eq. (46), the retarded correlation function $K^R(\omega)$ is obtained from the Matsubara correlator

$$K(\tau_{12}) = -\frac{g}{4}\alpha(\tau_{12})\langle e^{i[\varphi(\tau_1)-\varphi(\tau_2)]}\rangle \quad (51)$$

by standard analytic continuation. The physical observables $g'(T)$ and $q'(T)$ describe a response of the system to a change in the boundary condition Eq. (26). One-instanton contribution to the physical observables $g'$ and $q'$ reveals

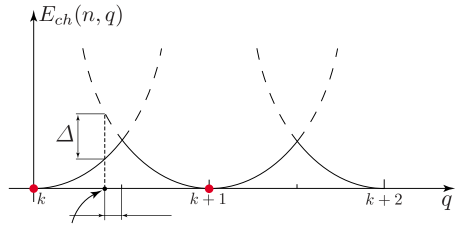


FIG. 3. (Color online) Charging energy $E_{\rm ch}=E_c(n-q)^2$ as a function of gate charge $q$.

their periodic dependence on the external charge $q$ as[24,36]

$$g'(T) = g(T)[1 - Dg(T)e^{-g(T)/2}\cos 2\pi q],$$

$$q'(T) = q - \frac{D}{4\pi}g^2(T)e^{-g(T)/2}\sin 2\pi q. \quad (52)$$

Some remarks on the physical meaning of these quantities are in order here. In the perturbative regime $g'(T)$ coincides with the renormalized coupling constant $g(T)$ while $q'(T)$ does not undergo any renormalization and coincides with the external charge, $q'(T)=q$. Thus, roughly speaking, we can think of them as the physical observables corresponding to the action parameters $g$ and $q$. The physics behind quantities Eq. (50) becomes even more pronounced if we turn from a SEB to a single-electron transistor (see Fig. 2). In the absence of dc voltage between left and right leads, a SET is described by the very same AES action Eqs. (23)–(27) in which the bare coupling constant $g=g_l+g_r$. Here, $g_{l/r}$ denotes the dimensionless conductances of the left/right tunneling junction. The quantity $g'(T)$ then coincides with the SET conductance[25,35] up to a temperature-independent factor

$$G(T) = \frac{e^2}{h}\frac{g_l g_r}{(g_l+g_r)^2}g'(T). \quad (53)$$

Expression for $q'(T)$ is possible to write in terms of antisymmetrized electron current-current correlator[24,36]

$$q'(T) = Q - i\frac{(g_l+g_r)^2}{2g_l g_r}\frac{\partial}{\partial V_{\rm dc}}\int_{-\infty}^{0}\langle[\hat{I}(0),\hat{I}(t)]\rangle|_{V_{\rm dc}=0}, \quad (54)$$

where $V_{\rm dc}$ denotes the dc voltage between the left and the right leads and $\hat{I}(t)=d\hat{n}_d(t)/dt$ the current operator for the SET.

For reasons to be explained shortly, it is natural to define the renormalized gate capacitance

$$C_g(T) = \frac{\partial q'(T)}{\partial U_0}. \quad (55)$$

According to Eq. (50), the quantity $C_g(T)$ is different from the effective capacitance $\partial Q/\partial U_0$ which has been considered in the literature so far. On the perturbative level $C_g(T)$ coincides with $C_g$; only instanton effects make it temperature and gate-voltage dependent,

$$C_g(T) = C_g\left[1 - \frac{D}{2}g^2(T)e^{-g(T)/2}\cos 2\pi q\right]. \quad (56)$$

Now, as usual, we plug in the bare capacitance $C_g$ expressed via $C_g(T)$ into Eq. (49). We see that instanton corrections cancel each other and the result (49) for the function $\mathcal{A}$ which determines the energy-dissipation rate becomes

$$\mathcal{A}(T) = \frac{2\pi C_g^2(T)}{g(T)}. \quad (57)$$

With the same level accuracy we can substitute $g'(T)$ for $g(T)$ and obtain finally the following expressions for the energy-dissipation rate and the admittance in the quasistatic regime:

$$\mathcal{W}_\omega = \frac{1}{2}\omega^2 C_g^2(T)R_q(T)|U_\omega|^2, \quad R_q(T) = \frac{h}{e^2 g'(T)}, \quad (58)$$

$$\mathcal{G}(\omega) = -i\omega\frac{\partial Q}{\partial U_0} + \frac{C}{C_g}C_g^2(T)R_q(T)\omega^2. \quad (59)$$

Several remarks are in order here. The results (58) and (59) are valid in the weak-coupling regime, $g'(T)\gg 1$, in which the quantities $\partial Q/\partial U_0$, $g'(T)$, and $C_g(T)$ are given by Eqs. (47), (52), and (56), respectively. Relations (58) and (59) fully describe the quasistatic dynamics of SEB. The energy-dissipation rate factorizes into the product of well-defined physical observables in complete analogy with classical expression (1). The admittance behavior is different from what we were expecting to get. Indeed, its imaginary and real components involve two different capacitances: effective capacitance $\partial Q/\partial U_0$ and renormalized gate capacitance $C_g(T)$. Moreover, the temperature-independent factor $C/C_g$ survives in the real part of $\mathcal{G}(\omega)$.

## IV. STRONG-COUPLING REGIME, $g\ll 1$

As follows from Eq. (50), the physical observables $g'(T)$ and $q'(T)$ are defined for arbitrary values of $g$. Therefore, it is of great interest to compute energy-dissipation rate and the SEB admittance in the opposite regime, of small dimensionless tunneling conductance $g\ll 1$. The question we ask is whether the results (58) and (59) with the proper $C_g(T)$ and $R_q(T)$ hold? We mention that the case $g\ll 1$ is a strong-coupling regime from the field-theoretical point of view. In what follows, we compute energy-dissipation rate by means of two different approaches. The first one is a refined field-theoretical method centered around Matveev's projective Hamiltonian.[13] The second one is more straightforward approach of rate equations on which the "orthodox theory" of Coulomb blockade was based.[37] We demonstrate how these two approaches beautifully complement each other.

### A. Preliminaries

We center our effort around the most interesting case; the vicinity of a degeneracy point, $q=k+1/2$, where $k$ is an integer (see Fig. 3). Following Ref. 13, the Hamiltonian (3)–(6) can be simplified by truncating the Hilbert space of electrons on the island to two charging states: with $Q=k$ and $Q=k+1$. The projected Hamiltonian then takes a form of $2\times 2$ matrix acting in the space of these two charging states. Denoting the deviation of the external charge from the degeneracy point by $\Delta$: $q=k+1/2-\Delta/(2E_c)$ we write the projected hamiltonian as[13]

$$H = H_0 + H_t + \Delta S_z + \frac{\Delta^2}{4E_c} + \frac{E_c}{4}, \tag{60}$$

where $H_0$ is given by Eq. (4) and

$$H_t = \sum_{k,\alpha} t_{k\alpha} a_k^\dagger d_\alpha S^+ + \text{H.c.} \tag{61}$$

Here, $S^z, S^\pm = S^x \pm iS^y$ are ordinary (iso)spin-1/2 operators. The presence of small ac component in the gate voltage changes the parameter $\Delta$ according to $\Delta \to \Delta - (eC_g/C)U_\omega \cos \omega t$. This time the response of the system to ac gate voltage is determined by the isospin correlation function $\Pi_s^R(\omega)$ (see Appendix A) which Matsubara counterpart is given by

$$\Pi_s(\tau) = \langle \mathcal{T}_\tau S^z(\tau) S^z(0) \rangle. \tag{62}$$

The energy-dissipation rate and SEB admittance can be expressed as follows:

$$\mathcal{W}_\omega = \frac{C_g^2}{2C^2}\omega \,\text{Im}\, \Pi_s^R(\omega) |U_\omega|^2,$$

$$\mathcal{G}(\omega) = -i\omega \frac{C_g}{C} \Pi_s^R(\omega). \tag{63}$$

Therefore, we need to proceed with the computation of $\Pi_s(\tau)$.

To deal with spin operators it is convenient to use Abrikosov's pseudofermion technique.[38] We introduce two-component pseudofermion operators $\psi_\alpha^\dagger$ and $\psi_\alpha$ such that

$$S^i = \psi_\alpha^\dagger S^i_{\alpha\beta} \psi_\beta. \tag{64}$$

Pseudofermions bring in the redundant unphysical states when $\sum_\alpha \psi_\alpha^\dagger \psi_\alpha > 1$. To exclude these states one adds an additional chemical potential $\eta$ to the Hamiltonian. It is necessary to set $\eta \to -\infty$ at the end of any calculation. The physical partition function $\mathcal{Z}$ and correlators $\langle \mathcal{O} \rangle$ can be found from the pseudofermionic ones as

$$\mathcal{Z} = \lim_{\eta \to -\infty} \frac{\partial}{\partial e^{\beta\eta}} \mathcal{Z}_{\text{pf}},$$

$$\langle \mathcal{O} \rangle = \lim_{\eta \to -\infty} \left\{ \langle \mathcal{O} \rangle_{\text{pf}} + \frac{\mathcal{Z}_{\text{pf}}}{\mathcal{Z}} \frac{\partial}{\partial e^{\beta\eta}} \langle \mathcal{O} \rangle_{\text{pf}} \right\}. \tag{65}$$

The elegance of pseudofermion technique lies in the fact that diagrams with pseudofermion loops vanish when one sets $\eta \to -\infty$.

Next, we plug representation [Eq. (64)] into Hamiltonian (60), switch to Matsubara basis and integrate out electrons in the lead and the island. Done in the parametric regime Eq. (11) this leads to the following effective action:

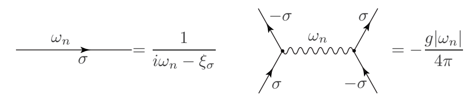


FIG. 4. Feynman rules for pseudofermion action; $\xi_\sigma = -\eta + \frac{\sigma\Delta}{2}$.

$$S = \int_0^\beta d\tau \bar{\psi} \left( \partial_\tau + \frac{\sigma_z \Delta}{2} - \eta \right) \psi + \frac{g}{4} \int_0^\beta d\tau_1 d\tau_2 \alpha(\tau_{12})$$
$$\times [\bar{\psi}(\tau_1)\sigma_- \psi(\tau_1)][\bar{\psi}(\tau_2)\sigma_+ \psi(\tau_2)] + \frac{\beta\Delta^2}{4E_c} + \frac{\beta E_c}{4}. \tag{66}$$

Here, $\sigma_i$ stand for Pauli matrices and $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$. Action similar to Eq. (66) has been first analyzed by Larkin and Melnikov in Ref. 39. In modern terminology, Eq. (66) corresponds to the XY case of the Bose-Kondo model for the spin 1/2.[40–42] Effective action [Eq. (66)] is very suitable for our purpose since it is coupling constant $g \ll 1$ justifying perturbative expansion.

First, we establish the relation between pseudofermion and physical partition function. From Eq. (65), we find

$$\mathcal{Z} = \lim_{\eta \to 0} \mathcal{Z}_{\text{pf}} e^{-\beta\eta} \sum_\sigma G_\sigma(\tau)|_{\tau \to 0^-} \tag{67}$$

Here, we denoted $G_\sigma(\tau) = -\langle \mathcal{T}_\tau \psi_\sigma(\tau) \bar{\psi}_\sigma(0) \rangle$ the exact pseudofermion Green's function. The Feynman rules for action [Eq. (66)] are shown in Fig. 4. In the zeroth order in $g$, we obtain

$$G_\sigma(i\varepsilon_n) = \frac{1}{i\varepsilon_n - \xi_\sigma}, \quad \mathcal{Z} = 2\cosh \frac{\beta\Delta}{2}, \quad \mathcal{Z}_{\text{pf}} = 1, \tag{68}$$

where $\varepsilon_n = \pi T(2n+1)$ and $\xi_\sigma = -\eta + \sigma\Delta/2$. Spin-spin correlation function Eq. (62) written in terms of pseudofermions becomes

$$\Pi_{s,\text{pf}}(\tau) = \frac{1}{4} \langle \mathcal{T}_\tau [\bar{\psi}(\tau)\sigma^z \psi(\tau)][\bar{\psi}(0)\sigma^z \psi(0)] \rangle, \tag{69}$$

where the average is taken with respect to action [Eq. (66)]. The physical correlation function is obtained from $\Pi_{s,\text{pf}}(i\omega_n)$ according to Eq. (65).

### B. Spin-spin correlaion function $\Pi^R_{s,\text{pf}}(\omega)$. First order in $g$.

We start by calculating the polarization operator Eq. (69) in the lowest possible order of perturbation theory. It happened that the first nontrivial contribution to $\Pi_{s,\text{pf}}(i\omega_n)$ came from the first-order perturbation theory. The relevant Feynman diagrams are depicted in Fig. 5.

The computation of $\Pi_{s,\text{pf}}(i\omega_n)$ is rather straightforward and is presented in the Appendix D. The result is

$$\Pi_{s,\text{pf}}(i\omega_n) = \frac{g}{4\pi^2} \frac{F^R(i\omega_n) + F^R(-i\omega_n)}{(i\omega_n)^2} e^{\beta\eta} \sinh \frac{\beta\Delta}{2}, \tag{70}$$

where $F^R(\omega)$ is a regular in the upper half plane of $\omega$ function

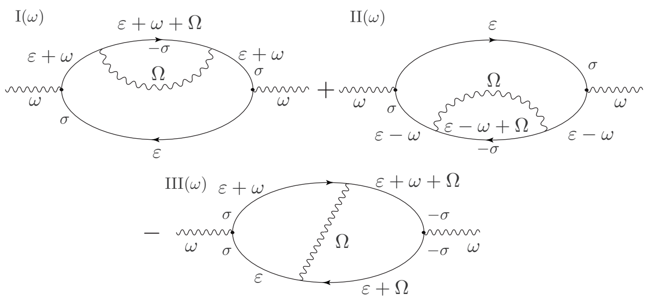


FIG. 5. Feynman diagrams defining the polarization operator in the lowest order.

$$F^R(\omega) = \sum_{\sigma=\pm 1}\left[(\Delta + \sigma\omega)\psi\left(\frac{\omega+\sigma\Delta}{2\pi Ti}\right) - \Delta\psi\left(\frac{i\sigma\Delta}{2\pi T}\right)\right]. \tag{71}$$

Here, $\psi(x)$ denotes the Euler digamma function. The analytical continuation should be made with some care. We want to recover the retarded polarization operator $\Pi_s^R(\omega)$ which is regular in the upper half plane of $\omega$. Since $\psi(x)$ has poles at $x_n = -n$ with natural $n$, the operator Eq. (70) has poles in both halves of a complex plane. We get rid of superfluous ones with the help of identity

$$\psi(z) - \psi(1-z) = -\pi \cot \pi z. \tag{72}$$

Using a well-known relation for real $x$

$$\text{Im } \psi(ix) = \frac{1}{2x} + \frac{\pi}{2}\coth \pi x \tag{73}$$

together with Eqs. (65) and (68), we arrive at the following expression for the imaginary part of the polarization operator:

$$\text{Im } \Pi_s^R(\omega) = \frac{g}{8\pi}\left\{\frac{\Delta}{\omega^2}\sum_{\sigma=\pm 1}\sigma \coth\frac{\Delta-\sigma\omega}{2T} + \frac{1}{\omega}\left[2\coth\frac{\Delta}{2T} - \sum_{\sigma=\pm 1}\coth\frac{\Delta-\sigma\omega}{2T}\right]\right\}\tanh\frac{\Delta}{2T}. \tag{74}$$

Expression (74) has a striking feature. It is divergent in the limit $\omega \to 0$. Indeed,

$$\text{Im } \Pi_s^R(\omega) = \frac{g}{4\pi\omega}\frac{\beta\Delta}{\sinh \beta\Delta}, \quad \omega \to 0. \tag{75}$$

The explanation is as follows. In essence the correlator Eq. (62) describes the noise of a fluctuating charge inside a metallic island. It was computed firstly in Ref. 43. The author of Ref. 43 however obtained a different (regular at $\omega \to 0$) expression. He used a special type of analytical continuation which yields a symmetric noise $\langle\{\hat{n}_d(t), \hat{n}_d(0)\}\rangle$. We, on the other hand, are interested in its antisymmetric counterpart which is the response function Eq. (16). It is exactly this retarded antisymmetric function which is obtained via standard analytical continuation procedure.

The unphysical divergency Eq. (75) comes from the nontrivial and essentially nonperturbative infrared structure of a polarization operator $\Pi_s^R(\omega)$. In what follows we prove that the partial summation of some infinite classes of diagrams resolves this singularity yielding the result:

$$\text{Im } \Pi_s^R(\omega) \sim \frac{g\omega}{z^2+\omega^2}, \tag{76}$$

where $z \sim g\Delta$ at $T=0$. As seen from Eq. (76) the limits $\omega \to 0$ and $g \to 0$ do not commute which explains how the artificial divergency in Eq. (75) arises. Now we proceed with a more accurate computation of the correlator $\Pi_s^R(\omega)$.

### C. One-loop structure of the pseudofermion theory

Throughout all our computation we will need some knowledge of the one-loop logarithmic structure of the pseudofermion theory. The bare Green's function is modified by the self-energy

$$G_\sigma(i\varepsilon_n) = \frac{1}{i\varepsilon_n - \xi_\sigma - \Sigma_\sigma(i\varepsilon_n)}. \tag{77}$$

The leading logarithmic approximation corresponds to one-loop renormalization. As it is known,[39] one extracts self-energy $\Sigma_\sigma(i\varepsilon_n)$ from a self-consistent Dyson equation

$$\Sigma_\sigma(i\varepsilon_n) = -\frac{g}{4\pi}T\sum_{\omega_m}|\omega_m|G_{-\sigma}(i\varepsilon_n + i\omega_m). \tag{78}$$

Here, we introduce $\omega_m = 2\pi Tm$. The vital observation[39,41,42] is that the action [Eq. (66)] can be renormalized with only one scaling parameter $Z$. Performing standard analytic continuation, we find[36,39]

$$G_\sigma^{R,A}(\varepsilon) = \frac{Z(\lambda)}{\varepsilon - \bar{\xi}_\sigma \pm i\bar{g}\Gamma_\sigma(\varepsilon)},$$

$$Z(\lambda) = \left(1 + \frac{g}{2\pi^2}\lambda\right)^{-1/2}, \quad \lambda = \ln\frac{E_c}{\max\{T, |\bar{\Delta}|, |\varepsilon|\}}. \tag{79}$$

Here, $\bar{\xi}_\sigma = -\eta + \sigma\bar{\Delta}/2$. $\bar{g} = gZ^2(\lambda)$ and $\bar{\Delta} = \Delta Z^2(\lambda)$ are the renormalized coupling constant and gap, respectively. As we see the energy $E_c$ plays the role of a reference energy scale. The important feature of Green's function Eq. (79) is its acquired width

$$\Gamma_\sigma(\varepsilon) = \frac{1}{8\pi}(\varepsilon - \bar{\xi}_{-\sigma})\frac{\cosh\frac{\varepsilon}{2T}}{\sinh\frac{\varepsilon-\bar{\xi}_{-\sigma}}{2T}\cosh\frac{\bar{\xi}_{-\sigma}}{2T}}. \tag{80}$$

According to the hierarchy of energy scales considered in the paper, $E_c \gg T$ and, therefore, the logarithmic corrections $\sim g \ln E_c/T$ are not small and require special care. To get read of large logarithms we change the reference scale of the field theory Eq. (66) from $E_c$ to $T$. With the help of result (79) we may rewrite the theory in terms of renormalized fields and running coupling constants: $\psi_\sigma \to \sqrt{Z(\lambda)}\psi_\sigma^r$, $g \to \bar{g}$, and $\Delta \to \bar{\Delta}$. The action then becomes

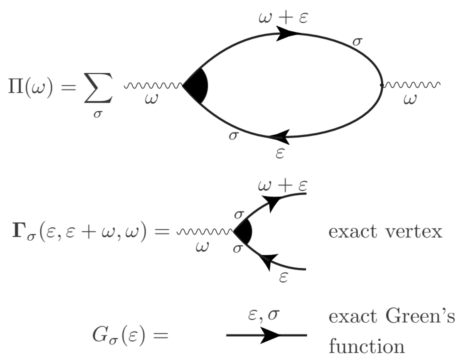


FIG. 6. Dyson equation for polarization operator $\Pi_{s,\mathrm{pf}}(i\omega_n)$.

$$S[\bar{\psi},\psi,\Delta,g] = S[\bar{\psi}^r,\psi^r,\bar{\Delta},\bar{g}] + \delta S_{\mathrm{ct}}, \tag{81}$$

where $\delta S_{\mathrm{ct}}$ stands for the action of counter terms. It is responsible for a consistent regularization of higher-order (in $\bar{g}$) corrections to the physical observables. Action Eq. (81) is very suitable for our purpose. All large logarithms are absorbed into coupling constants and fermionic fields. This allows us to drop counter terms in what follows. To bind the observables defined at the reference scale $E_c$ with the renormalized ones we shall need to establish scaling of the pseudofermion density $\rho_{\mathrm{pf}}=\Sigma_\sigma\langle\bar{\psi}_\sigma\psi_\sigma\rangle$ and $z$ component of the total spin density $s^z_{\mathrm{pf}}=(1/2)\Sigma_\sigma\sigma\langle\bar{\psi}_\sigma\psi_\sigma\rangle$. As follows from Ref. 39, the pseudofermion density $\rho$ has no scaling dimension of its own,

$$\rho_{\mathrm{pf}} = \sum_\sigma \langle\bar{\psi}^r_\sigma\psi^r_\sigma\rangle, \tag{82}$$

where now the average is taken with respect to action [Eq. (81)]. The total spin density $s_z$ has the same structure as the term proportional to $\Delta$ in action [Eq. (66)]. Therefore, it must have the same scaling dimension

$$s^z_{\mathrm{pf}} = \frac{1}{2}Z^2(\lambda)\sum_\sigma \sigma\langle\bar{\psi}^r_\sigma\psi^r_\sigma\rangle, \tag{83}$$

where again the average is taken with respect to the action [Eq. (81)]. For completeness we present the rigorous proof of Eq. (83) via Callan-Symanzik (CS) equation in Appendix D.

### D. Dyson equation for the spin-spin correlation function $\Pi^R_{s,\mathrm{pf}}(\omega)$.

The graphical representation of Dyson equation for the spin-spin correlation function

$$\Pi_{s,\mathrm{pf}}(i\omega_n) = \frac{T}{4}\sum_{\varepsilon_k,\sigma}\Gamma_\sigma(i\varepsilon_k,i\varepsilon_k+i\omega_n,i\omega_n) \times G_\sigma(i\varepsilon_k)G_\sigma(i\varepsilon_k+i\omega_n) \tag{84}$$

is illustrated in Fig. 6. Here, $\Gamma_\sigma(i\varepsilon_k,i\varepsilon_k+i\omega_n,i\omega_n)$ denotes the exact vertex function. Performing the analytic continuation in the spirit of Ref. 44, we find (see Appendix D for details)

$$\Pi^R_{s,\mathrm{pf}}(\omega) = -\sum_\sigma\int\frac{d\varepsilon}{16\pi i}\left\{\Gamma^{\mathrm{ARR}}_\sigma(\varepsilon,\varepsilon+\omega,\omega)G^A_\sigma(\varepsilon)G^R_\sigma(\varepsilon+\omega)\times\left[\tanh\frac{\varepsilon+\omega}{2T}-\tanh\frac{\varepsilon}{2T}\right]+\Gamma^{\mathrm{RRR}}_\sigma(\varepsilon,\varepsilon+\omega,\omega)\times G^R_\sigma(\varepsilon)G^R_\sigma(\varepsilon+\omega)\tanh\frac{\varepsilon}{2T}-\Gamma^{\mathrm{AAR}}_\sigma(\varepsilon,\varepsilon+\omega,\omega)\times G^A_\sigma(\varepsilon)G^A_\sigma(\varepsilon+\omega)\tanh\frac{\varepsilon+\omega}{2T}\right\}. \tag{85}$$

The most important task is to single out singular at $\omega\to 0$ and $\bar{g}\to 0$ terms in Eq. (85). We shall treat them separately to avoid divergencies. Firstly, we recall that we are interested in the quasistatic limit. Therefore, we shall proceed under assumptions $\omega\ll\max\{\bar{\Delta},T\}$ but $\omega\sim\bar{g}\max\{\bar{\Delta},T\}$. It is intuitively clear that a singular contribution always comes from the first term in the rhs of Eq. (85) which involves the product $G^A_\sigma G^R_\sigma$. Indeed, we observe that the pole structure of $G^A_\sigma G^R_\sigma$ always leads to a singular denominator of the type $(\omega+2i\bar{g}\Gamma_\sigma)$ as a result of integration. This happens due to the proximity of poles in $G^R_\sigma$ and $G^A_\sigma$. In contrast, the other terms with $G^R_\sigma G^R_\sigma$ and $G^A_\sigma G^A_\sigma$ are regular at $\bar{g}=\omega=0$ and therefore, free of divergencies. We may compute their contribution setting $\bar{g}=0$ and safely expanding the result in $\omega$. The integrand in Eq. (85) also has a series of Matsubara-type poles due to the presence of hyperbolic functions. These poles lead to logarithmically divergent sums. The latter are controlled by the renormalization scheme. In our case all leading logarithms are absent. They have already been absorbed into renormalized constants $\bar{g}$ and $\bar{\Delta}$ by the proper choice of reference energy scale. Thus we can drop all divergent sums over Matsubara frequencies.

Perfoming integration over $\varepsilon$ in Eq. (85) and expanding in $\omega$, where it is allowed, we are able to write down a much simpler expression for $\Pi^R_{s,\mathrm{pf}}(\omega)$,

$$\Pi^R_{s,\mathrm{pf}}(\omega) = \sum_\sigma\frac{\beta}{16\cosh^2\frac{\bar{\xi}_\sigma}{2T}}\left\{1-\frac{\omega\Gamma^{\mathrm{ARR}}_\sigma(\bar{\xi}_\sigma,\bar{\xi}_\sigma+\omega,\omega)}{\omega+2i\bar{g}\Gamma_\sigma(\bar{\xi}_\sigma)}\right\}. \tag{86}$$

Now we need to compute the vertex function $\Gamma^{\mathrm{ARR}}_\sigma(\varepsilon,\varepsilon+\omega,\omega)$. The vertex function $\Gamma_\sigma(i\varepsilon_k,i\varepsilon_k+i\omega_n,i\omega_n)$ satisfies Dyson equation

$$\Gamma_\sigma(i\varepsilon_k,i\varepsilon_k+i\omega_n,i\omega_n) = 1+\frac{\bar{g}T}{4\pi}\sum_{\omega_m}|\omega_m|G_{-\sigma}(i\varepsilon_k+i\omega_m)\times G_{-\sigma}(i\varepsilon_k+i\omega_m+i\omega_n)\times\Gamma_{-\sigma}(i\varepsilon_k+i\omega_m,i\varepsilon_k+i\omega_m+i\omega_n,i\omega_n) \tag{87}$$

which is shown in Fig. 7.

The details of analytical continuation are described in Appendix D where we prove that the Dyson equation for the vertex $\Gamma^{\mathrm{ARR}}_\sigma(\varepsilon,\varepsilon+\omega,\omega)$ becomes

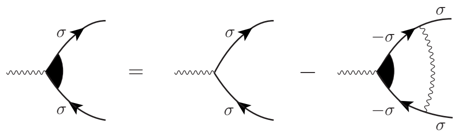


FIG. 7. Dyson equation for the vertex $\Gamma_\sigma$.

$$\Gamma_\sigma^{\mathrm{ARR}}(\varepsilon,\varepsilon+\omega,\omega) = 1 - \frac{\bar{g}}{8\pi}\int \frac{dx}{2\pi}\Gamma_{-\sigma}^{\mathrm{ARR}}(x,x+\omega,\omega) \times G_{-\sigma}^{A}(x)G_{-\sigma}^{R}(x+\omega)(x-\varepsilon) \times \left[2\coth\frac{x-\varepsilon}{2T} - \tanh\frac{x+\omega}{2T} - \tanh\frac{x}{2T}\right]. \tag{88}$$

To solve it we have to make some self-consistent guess. The apparent difficulty is that apart from singular factor $G_\sigma^R G_\sigma^A$ the integrand in the rhs of Eq. (88) may have unknown poles coming from the vertex function $\Gamma_{-\sigma}^{\mathrm{ARR}}$. We conjecture that these poles result in the contribution of order of unity and are small comparing to singular contribution from the product $G_\sigma^R G_\sigma^A$. Hence we may perform an integral in the rhs of Eq. (88) and arrive at the following result:

$$\Gamma_\sigma^{\mathrm{ARR}}(\varepsilon,\varepsilon+\omega,\omega) = 1 - \frac{\bar{g}}{8\pi}\frac{\bar{\xi}_{-\sigma}-\varepsilon}{\sinh\frac{\bar{\xi}_{-\sigma}-\varepsilon}{2T}} \times \left\{\frac{\cosh\frac{\varepsilon}{2T}}{\cosh\frac{\bar{\xi}_{-\sigma}}{2T}} + \frac{\cosh\frac{\varepsilon+\omega}{2T}}{\cosh\frac{\bar{\xi}_{-\sigma}+\omega}{2T}}\right\} \times \frac{\Gamma_{-\sigma}^{\mathrm{ARR}}(\bar{\xi}_{-\sigma},\bar{\xi}_{-\sigma}+\omega,\omega)}{-i\omega+2\bar{g}\Gamma_{-\sigma}(\bar{\xi}_{-\sigma})}. \tag{89}$$

We see that the solution does have an additional series of poles in the $\varepsilon$ plane. However these are Matsubara-type poles and are irrelevant as was argued above. Setting the external energy $\varepsilon=\bar{\xi}_\sigma$ we obtain the self-consistent equation on $\Gamma_\sigma^{\mathrm{ARR}}(\bar{\xi}_\sigma,\bar{\xi}_\sigma+\omega,\omega)$. The solution reads

$$\Gamma_\sigma^{\mathrm{ARR}}(\bar{\xi}_\sigma,\bar{\xi}_\sigma+\omega,\omega) = \frac{1}{\omega}\frac{[\omega+2i\bar{g}(\Gamma_{-\sigma}-\Gamma_\sigma)][\omega+2i\bar{g}\Gamma_\sigma]}{\omega+2i\bar{g}(\Gamma_{-\sigma}+\Gamma_\sigma)} \tag{90}$$

Here, $\Gamma_\sigma=\Gamma_\sigma(\bar{\xi}_\sigma)$ is the width of the Green's function defined in Eq. (80).

Collecting Eqs. (86) and (89) we write down the result for the spin-spin correlation function

$$\Pi_{s,\mathrm{pf}}^R(\omega) = \frac{\bar{g}}{4\pi}\frac{\bar{\Delta}}{T\sinh\frac{\bar{\Delta}}{2T}}\left[-i\omega+\frac{\bar{g}\bar{\Delta}}{2\pi}\coth\frac{\bar{\Delta}}{2T}\right]^{-1}. \tag{91}$$

Finally, taking into account Eqs. (67) and (68), we obtain the following result for the spin-spin correlator:

$$\Pi_s^R(\omega) = \frac{\bar{g}Z^4}{4\pi}\frac{\bar{\Delta}}{T\sinh\frac{\bar{\Delta}}{T}}\left[-i\omega+\frac{\bar{g}\bar{\Delta}}{2\pi}\coth\frac{\bar{\Delta}}{2T}\right]^{-1}. \tag{92}$$

Here we restored factor $Z^4$ which provides a correct scaling dimension of spin fields according to Eq. (83).

### E. Admittance and energy-dissipation rate

With the help of Eq. (63) we obtain the admittance of the SEB for frequencies $\omega \ll \max\{\bar{\Delta},T\}$

$$\mathcal{G}(\omega) = \frac{C_g}{C}\frac{\bar{g}Z^4}{4\pi}\frac{\bar{\Delta}}{T\sinh\frac{\bar{\Delta}}{T}}\frac{-i\omega}{-i\omega+\frac{\bar{g}\bar{\Delta}}{2\pi}\coth\frac{\bar{\Delta}}{2T}}. \tag{93}$$

The average charge $Q$ and the physical observables $g'(T)$ and $q'(T)$ can be found from the pseudofermion theory Eq. (66) if one substitutes the transverse spin-spin correlation function

$$K_s(\tau_{12}) = -\frac{g}{4}\alpha(\tau_{12})\langle S^+(\tau_1)S^-(\tau_2)\rangle \tag{94}$$

for $K(\tau_{12})$ in Eq. (50).[24,36] In the leading logarithmic approximation, the average charge and the physical observable $g'$ are given by[45]

$$Q(T) = k + \frac{1}{2} - \frac{Z^2}{2}\tanh\frac{\bar{\Delta}}{2T}, \tag{95}$$

$$g'(T) = \frac{\bar{g}}{2}\frac{\bar{\Delta}}{T\sinh\frac{\bar{\Delta}}{T}}. \tag{96}$$

The temperature dependence of the other physical observable $q'$ is as follows:[36]

$$q'(T) = k + \frac{1}{2} - \frac{1}{2}\tanh\frac{\bar{\Delta}}{2T}. \tag{97}$$

To get the energy-dissipation rate we expand expression (93) in $\omega$. Using the identity $d\bar{\Delta}=-Z^2 dU_0/C$ and Eqs. (95)–(97), we obtain the energy-dissipation rate and the admittance of the SET in the quasistatic regime

$$\mathcal{W}_\omega = \frac{1}{2}\omega^2 C_g^2(T)R_q(T)|U_\omega|^2, \quad R_q(T) = \frac{h}{e^2 g'(T)}, \tag{98}$$

$$\mathcal{G}(\omega) = -i\omega\frac{\partial Q}{\partial U_0} + \frac{C}{C_g}C_g^2(T)R_q(T)\omega^2. \quad (99)$$

Here, the renormalized gate capacitance and the effective capacitance becomes

$$C_g(T) = \frac{\partial q'}{\partial U_0} = C_g\frac{Z^2}{2}\frac{E_c}{T\cosh^2\frac{\bar{\Delta}}{2T}}, \quad (100)$$

$$\frac{\partial Q}{\partial U_0} = C_g\frac{Z^4}{2}\frac{E_c}{T\cosh^2\frac{\bar{\Delta}}{2T}}. \quad (101)$$

Several remarks are in order here. The results (98) and (99) are valid in the strong-coupling regime, $g \ll 1$ and near the degeneracy point $|\Delta| \ll E_c$. The accuracy we are working with (the leading logarithmic approximation) allows us to make the following key observation. The expressions for energy-dissipation rate Eq. (98) and admittance Eq. (99) cast in terms of the quantities $\partial Q/\partial U_0$, $g'(T)$, and $C_g(T)$ coincide with the ones obtained in the weak-coupling regime. It makes us suggest that the results (98) and (99) are valid for all temperature range $E_c \gg T \gg \delta$ and all values of $g$.

We mention that formula (98) for $R_q(T)$ is a truly nonperturbative in $g$ result. Despite obvious complications we overcame to obtain it, the expression for $g'(T)$ (stripped of all logarithmic scaling) is the same as obtained in a much simpler approach of sequential tunneling. This approach known as the "orthodox theory" of a Coulomb blockade will help us to shed light on the physical meaning of results (98) and (99). Further calculations are formulated in the language of rate equations[37] which lie in the basis of the orthodox theory.

### F. Rate-equations approach

The rate approach is less general since it is essentially a Fermi golden rule approximation. It overlooks virtual processes and is unable to reproduce logarithmic scaling of physical observables. On the other hand rate equations are much easier to solve than corresponding Dyson equations used above in a field-theoretical treatment. We are going to demonstrate that rate equations allow us to find the admittance for frequencies which are not restricted by the condition $\omega \ll \max\{\bar{\Delta}, T\}$ imposed by the field approach. Eventually we will conceive a prescription on how the admittance formula (93) can be generalized for arbitrary (but still not very large $\omega \ll E_c$) frequencies.

As above, only two charging states of the island are counted. We denote them as follows: state 0 corresponds to the average charge $Q=k$ and state 1 corresponds to $Q=k+1$. Probabilities for each state are denoted as $p_0$ and $p_1$ which satisfy $p_0+p_1=1$. Master equation has the standard form[37]

$$\dot{p}_0 = -\Gamma_{10}p_0 + \Gamma_{01}p_1. \quad (102)$$

Here, $\Gamma_{01}$ and $\Gamma_{10}$ are tunneling rates *from* and *to* the metallic island, respectively. We should keep in mind that tunneling rates $\Gamma_{01/10}$ are proportional to dimensionless conductance of the tunneling junction $g$ which is the expansion parameter of our problem. The average current through the contact is $I=-\dot{p}_0$. Since we are interested in the linear response of the current to the time-dependent gate voltage $U(t)=U_0+U_\omega\cos\omega t$, we expand the tunneling rates to the first order in amplitude $U_\omega$

$$\Gamma_{01/10}(t) = \Gamma^0_{01/10} + \frac{C_gU_\omega}{2C}[\gamma_{01/10}(\omega)e^{-i\omega t} + \gamma_{01/10}(-\omega)e^{i\omega t}]. \quad (103)$$

Then it is easy to find the following relation for the admittance:

$$\mathcal{G}(\omega) = -i\omega\frac{C_g}{C}\frac{\gamma_{10}(\omega)\Gamma^0_{01} - \gamma_{01}(\omega)\Gamma^0_{10}}{(\Gamma^0_{01}+\Gamma^0_{10})(-i\omega+\Gamma^0_{01}+\Gamma^0_{10})}. \quad (104)$$

The equilibrium tunneling rates are well known[37]

$$\Gamma^0_{01/10} = \frac{g\Delta}{4\pi}\left(\coth\frac{\Delta}{2T} \pm 1\right). \quad (105)$$

We mention that up to the logarithmic corrections $\Gamma^0_{01/10}=2g\Gamma_\pm$. A straightforward calculation of the tunneling rates yields (see Appendix E)

$$\gamma_{01/10}(\omega) = \mp\frac{g}{4\pi}\left[1 \pm \frac{i}{\pi\omega}F^R(\omega)\right], \quad (106)$$

where function $F_R(\omega)$ was introduced in Eq. (71). Plugging Eqs. (105) and (106) into Eq. (104) we arrive at the general expression for admittance

$$\mathcal{G}(\omega) = \frac{C_g}{C}\frac{g}{4\pi\coth\frac{\beta\Delta}{2}}\frac{-i\omega\coth\frac{\Delta}{2T} - \frac{1}{\pi}F^R(\omega)}{-i\omega + \frac{g\Delta}{2\pi}\coth\frac{\Delta}{2T}}. \quad (107)$$

In order to relate this result to field-theoretical result (93), we expand the function $F^R(\omega)$ to the first order in $\omega$,

$$F^R(\omega) = i\pi\omega\left(\frac{\Delta}{2T\sinh^2\frac{\Delta}{2T}} - \coth\frac{\Delta}{2T}\right) + \mathcal{O}(\omega^2). \quad (108)$$

Plugging this into Eq. (104) we get the familiar expression

$$\mathcal{G}(\omega) = \frac{C_g}{C}\frac{g}{4\pi}\frac{\beta\Delta}{\sinh\beta\Delta}\frac{-i\omega}{-i\omega + \frac{g\Delta}{2\pi}\coth\frac{\Delta}{2T}} \quad (109)$$

which is valid for $\omega \ll \max\{\Delta, T\}$ and nearly repeats result (93) for the admittance. The only difference is in the scaling factor $Z$ which is absent in the rate-equations approach. Now we may easily guess a prescription on how to generalize Eq. (93) for arbitrary $\omega$. A correctly defined observable admittance ought to scale as $Z^4$. It should also be expressed in terms of the renormalized parameters $\bar{g}$ and $\bar{\Delta}$ only. This leads us to the following result:

$$\mathcal{G}(\omega) = \frac{C_g}{C} \frac{\bar{g}Z^4}{4\pi \coth\frac{\bar{\Delta}}{2T}} \frac{-i\omega \coth\frac{\bar{\Delta}}{2T} + \frac{1}{\pi}\bar{F}^R(\omega)}{-i\omega + \frac{\bar{g}\bar{\Delta}}{2\pi}\coth\frac{\bar{\Delta}}{2T}} \quad (110)$$

which as we believe describes the admittance for $\omega \ll E_c$ in the strong-coupling regime $g \ll 1$. Here, the function $\bar{F}^R(\omega)$ is given by $F_R(\omega)$ in which $\bar{\Delta}$ is substituted for $\Delta$. Finally, we mention that at finite frequency $\omega$ the parameter $\lambda$ which determines the scaling parameter $Z$ in Eq. (79) should be modified as follows:

$$\lambda = \ln \frac{E_c}{\max\{T, |\bar{\Delta}|, |\omega|\}}. \quad (111)$$

## V. DISCUSSIONS AND CONCLUSIONS

As we demonstrated in the previous sections the energy-dissipation rate $\mathcal{W}_\omega$ is given by Eq. (2) with $R_q(T) = h/e^2 g'(T)$ and $C_g(T) = \partial q'(T)/\partial U_0$ in both weak- and strong-coupling regimes. We emphasize that the physical observables $g'$ and $q'$ are defined in terms of the correlation function of the phase field $\varphi(\tau)$ of the AES model [see Eq. (50)]. Therefore, they can be found, in general, not only in weak- and strong-coupling regimes but for arbitrary values of $g$ and $q$. Hence, it is natural to assume that Eq. (2) as well as Eq. (59) hold, in general, for a SEB under conditions of applicability of the AES model which are $E_{\text{Th}} \gtrsim E_c \gg T \gg \delta \max\{1, g\}$ and $g/N_{\text{ch}} \ll 1$. Although, at present we are not able to prove this conjecture we believe strongly it is true.

Originally,[24] the physical quantity $q'(T)$ has been introduced for a SET and its physical meaning was interpreted in terms of the average charge on the island and the antisymmetrized current-current correlation function [see Eq. (54)]. If we introduce nonsymmetrized current noise in the SET (Refs. 46 and 47)

$$S_I(\omega, V_{\text{dc}}) = \int_{-\infty}^{\infty} dt e^{-i\omega t} \langle \hat{I}(t)\hat{I}(0)\rangle, \quad (112)$$

then we can present Eq. (54) as

$$q' = Q + \frac{(g_l + g_r)^2}{2\pi g_l g_r} \text{p.v.} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{\partial S_I(\omega, V_{\text{dc}})}{\partial V_{\text{dc}}}\Bigg|_{V_{\text{dc}}=0}. \quad (113)$$

Therefore, to measure $q'(T)$ two separate experiments are needed: the measurement of the average charge on the island at $V_{\text{dc}}=0$ and the measurement of the nonsymmetrized current noise $S_I(\omega, V_{\text{dc}})$. Although experimental designs probing the nonsymmetrized current noise have already been proposed[48] and measurements have recently been taken from a number of electronic quantum devices,[49] it is still a challenge. Our present results indicate that the quantity $q'$ can be related to the renormalized gate capacitance $C_g(T)$. Namely, $C_g(T) = \partial q'/\partial U_0$, provided that result (2) holds in general (not in a weak- and strong-coupling regimes only). The latter capacitance can be extracted from measurements of the energy-dissipation rate and the SET conductance.

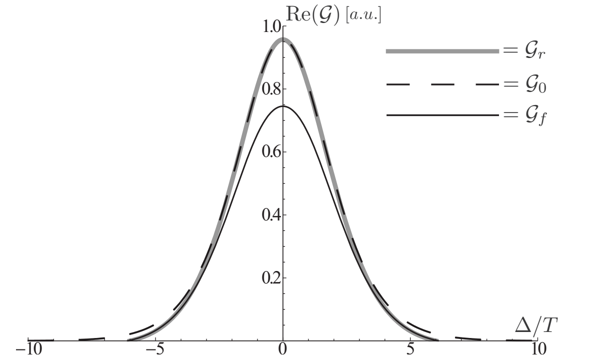


FIG. 8. The dissipative part of admittance of the SEB at fixed $\omega$ as a function of $\Delta$. Three plots using three different formulae are presented. $G_r$ is given by Eq. (107), $G_0$ is given by Eq. (109), and $G_f$ is given by Eq. (110). We use $g=0.5$, $E_c=10T$, and $\omega=0.8T$.

Recently, the energy-dissipation rate of SEB has been addressed experimentally via radio-frequency reflectometry measurements (by sending a continuous radio-frequency signal to the device).[12] The temperature and external charge dependences of the quantity $[\omega^2 \mathcal{A}(T)]^{-1}$ were studied. The latter quantity was referred to as the "Sisyphus" resistance by the authors of Ref. 12. In the experiment the tunneling conductance was estimated to be equal $g \approx 0.5$ such that the SEB was in the strong-coupling regime. In Ref. 12 the Sisyphus resistance was estimated theoretically within the rate-equation approach [see Eq. (4) in Ref. 12]. Their result corresponds to our result (109) for the admittance. However, our final result for the admittance (110) is more general then Eq. (4) of Ref. 12. The latter does not take into account not only the logarithmic renormalizations of the SEB parameters but also deviation of the function $F^R(\omega)$ from the linear one. Although the values of the SEB parameters reported in Ref. 12 are such that the difference of the scaling factor $Z$ from unity is several percent, logarithmic renormalizations in the expression for the admittance yield noticable effect. This is shown in Fig. 8. In addition, the function $F^R(\omega)$ can be written as the linear one only for frequencies $\omega \ll \max\{\Delta, T\}$ which is not the case for the low-temperature data of Ref. 12. Therefore, the experimental data of Ref. 12 needs to be reanalyzed with the help of Eq. (110).

The authors of Ref. 12 claim that their results for the Sysiphus resistance indicate the violation of the Kirchhoff's laws. They argue that the admittance they measure does not correspond to the equivalent circuit of SEB with bare values of the gate capacitance $C_g$ and the tunneling conductance $g$. However, by the same logic one could claim the violation of the Kirchhoff's laws in measurements of the SET conductance $\mathcal{G}(T)$ because it is different from $g_l g_r/(g_l+g_r)$. Our results imply that the energy-dissipation rate (inverse of the Sysiphus resistance) in the SET can be obtained from the Kirchhoff's laws if one substitute $C_g$ and $g$ for $C_g(T)$ and $g'(T)$ in the equivalent circuit.

As one can see from Fig. 8, the energy-dissipation rate is maximal for $\Delta=0$ which corresponds to the half-integer values of the external charge $q$. It occurs because the larger the

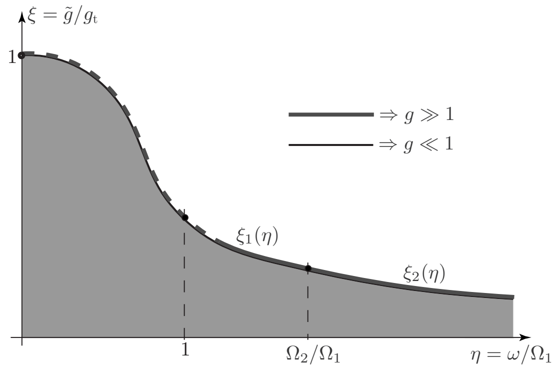


FIG. 9. Schematic for comparison of quantum and classical mechanisms of the energy dissipation. The quantum dissipation dominates in the filled region.

value of $\Delta$, the larger the ratio $\Gamma^0_{01}/\Gamma^0_{10}$ becomes. We remind that $\Gamma^0_{01/10}$ is the transition rate from (to) the state with $Q=k+1$ to (from) the state with $Q=k$. The increase in $\Delta$ makes the system less probable to be excited in the state with $Q=k+1$ by the time-dependent gate voltage and therefore, reduces energy dissipation. Of course, this physical explanation is strongly based on the model of only two charging states involved. It is valid at $g \ll 1$. However, at $g'(T) \gg 1$ the energy-dissipation rate has the maximum at half-integer values of the external charge $q$ as well [see Eq. (49)]. This result cannot be explained by arguments based on the "orthodox" theory since there are no well-defined charging states in the weak-coupling regime.

The dissipation caused by the electron tunneling is not the only one that occurs in the setup. Intrinsic electron transitions inside the metallic island cause an additional internal energy loss. This mechanism is, in fact, the origin of metallic conductivity. This sort of dissipation ought to be mainly a classical effect. It corresponds to the radiation of energy by a metallic particle placed in the quasistationary electric field. The classical dissipation can be conveniently characterized by two limiting regimes: the low-frequency ohmic loss and high-frequency nonohmic radiation (skin effect),

$$W^c_\omega \sim \frac{\hbar}{g_t e^2} R^2 \omega^2 |U_\omega|^2, \quad \omega \ll \omega_0, \quad \omega_0 = \frac{E_c}{g_t \alpha^2 \hbar},$$

$$W^c_\omega \sim \frac{\hbar}{g_t e^2} R^2 \omega^2 \left(\frac{\omega}{\omega_0}\right)^{3/2} |U_\omega|^2, \quad \omega \gg \omega_0 \qquad (114)$$

Here, $\alpha = e^2/\hbar c$ is a fine-structure constant, $g_t e^2/\hbar$ is an internal (Thouless) conductance of the island, $R$ is characteristic size, and $\omega_0$ is the separating frequency. To elucidate the parametric conditions under which quantum dissipation $\mathcal{W}_\omega$ due to presence of the tunneling junction dominates over the classical one we make necessary estimates. Quantum dissipation can also be split into ohmic and nonohmic limiting regimes. The corresponding separating frequency is denoted as $\Omega$. We are concerned with simple estimates only and drop weak log corrections in all formulae for the quantum case. The results are most transparently explained via phase diagram which is presented in Fig. 9 supplemented by Tables I and II.

TABLE I. Description of functions for Fig. 9.

| | $\omega_0 \le \Omega$ | $\omega_0 > \Omega$ |
|---|---|---|
| $\Omega_1$ | $\omega_0$ | $\Omega$ |
| $\Omega_2$ | $\Omega$ | $\omega_0$ |
| $\xi_1$ | $1/\eta^{3/2}$ | $1/\eta^2$ |
| $\xi_2$ | $\frac{1}{\eta^{7/2}}(\frac{\Omega_2}{\Omega_1})^2$ | $\frac{1}{\eta^{7/2}}(\frac{\Omega_2}{\Omega_1})^{3/2}$ |

In the fully coherent case, the admittance of SEB was studied in Ref. 7 by means of the $S$-matrix formalism. It was shown that the SEB admittance can be presented in accordance with its classical appearance Eq. (22) but the definition of physical quantities comprising it becomes different. In Ref. 7, it was derived that the gate capacitance $C_g$ and the tunneling resistance $R$ should be substituted by the mesoscopic capacitance $C_\mu$ and the charge relaxation resistance $R_q$, respectively. However, according to our results, although being applicable in the fully incoherent case, the SEB admittance in the quasistationary regime involves two capacitances: the effective capacitance $\partial Q/\partial U_0$ which controls the imaginary part of $\mathcal{G}(\omega)$ and the renormalized capacitance $C_g(T)$ which together with $R_q(T)$ determines the temperature behavior of Re $\mathcal{G}(\omega)$. It is the effective capacitance that corresponds to the mesoscopic capacitance $C_\mu$. The appearance of the effective capacitance $\partial Q/\partial U_0$ in the imaginary part of the admittance is dictated by conservation of charge via the Ward identity Eq. (20). We expect that the SEB admittance should involve two physically different capacitances, in general. Recently, the SEB admittance was studied with the help of the $S$-matrix formalism in the incoherent case also.[10] In particular, it was predicted that in the fully incoherent regime and at low temperatures the charge relaxation resistance $R_q = h/(ge^2)$. It is at odds with our result that $R_q = h/[e^2 g'(T)]$ since at low temperatures $g'(T)$ can be very different from $g$ [see Eqs. (52) and (96)]. The reason behind this discrepancy is as follows. Coulomb interaction in Ref. 10 was accounted for on the level of classical equations of motion only, which was the conservation of charge. In the mean-time quantum fluctuations of charge are significant throughout all our analysis and there is no obvious justification to take them negligible.

To summarize, we have studied the energy dissipation in a single-electron box due to a slowly oscillating gate voltage. We focused on the regime of not very low temperatures when electron coherence can be neglected but quantum fluctuations of charge are strong due to Coulomb interaction. We considered cases of weak and strong coupling. In both cases we found that the energy-dissipation rate is given by the

TABLE II. Description of parameters for Fig. 9.

| | $\Omega$ | $\tilde{g}$ |
|---|---|---|
| $g \gg 1$ | $gE_c/\hbar$ | $g$ |
| $g \ll 1,\ \Delta \ll T$ | $gT/\hbar$ | $g(\frac{T}{E_c})^2$ |
| $g \ll 1,\ \Delta \gg T$ | $g\Delta/\hbar$ | $g\frac{\Delta T}{E_c^2}e^{\Delta/T}$ |

same expression involving two physical observables $g'(T)$ and $C_g(T)$. Our result for the energy-dissipation rate can be obtained from the SEB equivalent circuit if one substitutes $g'(T)$ and $C_g(T)$ for $g$ and $C_g$, respectively. We strongly believe that the universal expression we found for the energy-dissipation rate is valid for an arbitrary value of the tunneling conductance.


### ACKNOWLEDGMENTS

The authors are indebted to L. Glazman, Yu. Makhlin, A.M.M. Pruisken, A. Semenov, and M. Skvortsov for helpful discussions. The research was funded by RFBR (Grant Nos. 09-02-92474-MHKC, No. 06-02-16533, and No. 07-02-00998), the Council for grants of the Russian President (Grant No. 4445.2007.2), the Dynasty Foundation, the Program of RAS "Quantum Macrophysics," and CRDF.


## APPENDIX A: ENERGY DERIVATIVE

Here we relate dissipation in the system to various field correlators in weak- and strong-coupling regime.

### 1. Weak coupling, $g \gg 1$

We want to express correlator Eq. (16) through AES effective phase $\phi(\tau)$. We want to be rigorous and introduce Keldysh contour (see Fig. 10).

We split all the fields into upper and lower components $(\pm)$ in correspondence to the halves of the Keldysh contour. The action of the system is split as well $S=S_+-S_-$ and the partition function of the system reads $\mathcal{Z}=\int \mathcal{D}\varphi_\pm e^{iS[\varphi_\pm]}=1$. The average electron density is found as

$$\sum_\alpha \langle d_\alpha^\dagger d_\alpha\rangle = \frac{1}{2}\sum_\alpha \langle d_{\alpha+}^\dagger d_{\alpha+} + d_{\alpha-}^\dagger d_{\alpha-}\rangle = \frac{C}{2C_g}\left\langle \frac{\delta S}{\delta U_{g,q}}\right\rangle + C_g U_c. \tag{A1}$$

Here, we introduced classical and quantum components for bosonic fields

$$U_{g,c} = \frac{1}{2}(U_{g+}+U_{g-}), \quad U_{g,q} = \frac{1}{2}(U_{g+}-U_{g-}), \tag{A2}$$

and $U_g(t)=U_0+U_\omega \cos \omega t$. To get rid of quartic Coulomb terms we introduce Hubbard-Stratonovich boson fields, $V_+,V_-$ on each part of the contour and make fermion gauge transformation

$$d_{\alpha\sigma} \to d_{\alpha\sigma}e^{-i\int_0^t V_\sigma dt}. \tag{A3}$$

The transformed terms $S_0,S_c,S_t$ take the form

$$S_{0\sigma} = \sum_k \int_{-\infty}^{\infty} a_{k\sigma}^\dagger (i\partial_t - \varepsilon_k^{(a)})a_{k\sigma}dt + \sum_\alpha \int_{-\infty}^{\infty} d_{\alpha\sigma}^\dagger (i\partial_t - \varepsilon_\alpha^{(d)})d_{\alpha\sigma}dt,$$

$$S_{c\sigma} = \frac{C}{2}\int_{-\infty}^{\infty} V_\sigma^2 dt + C_g \int_{-\infty}^{\infty} V_\sigma U_{g,\sigma}(t)dt,$$

$$S_{t\sigma} = -\sum_{\alpha,k}\int_{-\infty}^{\infty}\{t_{k\alpha}e^{i\int V_\sigma dt}a_{k\sigma}^\dagger d_{\alpha\sigma} + \text{H.c.}\}dt. \tag{A4}$$

We see that the source term $U_{g,\sigma}$ enter $S_{c\sigma}$ only, hence we regroup it in a more suitable form

$$S_c = S_{c+} - S_{c-} = C\int_{-\infty}^{\infty} V_c V_q dt + \sqrt{2}C_g \int_{-\infty}^{\infty}(V_c U_{g,q} + V_q U_{g,c})dt. \tag{A5}$$

Here, $V_{c,q}=1/\sqrt{2}(V_+ \pm V_-)$. Next we find the physical electron density from Eq. (A1)

$$\sum_\alpha \langle d_\alpha^\dagger d_\alpha\rangle = \frac{C}{\sqrt{2}}\langle V_c\rangle + C_g U_{g,c},$$

$$\left\langle \frac{\partial H}{\partial U_g}\right\rangle = -\frac{C_g}{\sqrt{2}}\langle V_c\rangle. \tag{A6}$$

We then expand $e^{iS}$ to linear order in *physical* field $U_c$. The result reads

$$\left\langle \frac{\partial H}{\partial U_g}\right\rangle = -\frac{C_g^2}{C^2}\int_{-\infty}^{\infty}\Pi_R(t-t')U_g(t')dt', \tag{A7}$$

where $\Pi_R(t-t')=iC^2\langle V_c(t)V_q(t')\rangle$. Coupled with Eq. (13) it gives Eq. (28).

Using Eq. (A6) we also write down the formula for the effective capacitance $\partial Q/\partial U_0$ of the SEB,

$$\frac{\partial Q}{\partial U_0} = C_g + \frac{\Pi_R(0)}{C}. \tag{A8}$$

### 2. Strong coupling, $g \ll 1$

We proceed in complete analogy with the previous case. Using Hamiltonian (60) we obtain

$$\left\langle \frac{\partial H}{\partial U_g}\right\rangle = -\frac{C_g}{C}\langle S^z(t)\rangle. \tag{A9}$$

Keldysh technique gives

$$\langle S^z(t)\rangle = \frac{C_g}{C}\int_{-\infty}^{\infty} i\langle S_c^z(t)S_q^z(t')\rangle U_c(t')dt'. \tag{A10}$$

Introducing spin-correlation function

$$\Pi_s^R(t) = i\langle S_c^z(t)S_q^z(0)\rangle \tag{A11}$$

we recover dissipation expression (63) with spin correlator $\Pi_s^R(\omega)$ playing the role of polarization operator.

## APPENDIX B: ADMITTANCE

The admittance is defined as

$$\frac{\delta\langle I(t)\rangle}{\delta U_g(t')} = \int_{-\infty}^{\infty}\mathcal{G}(\omega)e^{-i\omega(t-t')}\frac{d\omega}{2\pi}. \tag{B1}$$

We introduce the tunneling current operator using Hamiltonian (3)–(6)

$$I = i\left[H, \sum_{\alpha} d_{\alpha}^{\dagger} d_{\alpha}\right] = i\sum_{k,\alpha} t_{k\alpha} a_k^{\dagger} d_{\alpha} + \text{H.c.} \tag{B2}$$

To find the average current we insert the necessary source term into the action

$$S_s = \frac{1}{2}\int_{-\infty}^{\infty} I(t)\kappa(t)dt, \quad \langle I(t)\rangle = \frac{1}{i}\left.\frac{\partial \mathcal{Z}[I]}{\partial \kappa(t)}\right|_{\kappa=0}. \tag{B3}$$

While taking a functional integral along Keldysh contour we keep a quantum component of $\kappa(t)$ field only. We make the usual rotation in the fermion basis

$$\psi_{\pm} = \frac{1}{\sqrt{2}}(\psi_1 \pm \psi_2),$$

$$\bar{\psi}_{\pm} = \frac{1}{\sqrt{2}}(\bar{\psi}_2 \pm \bar{\psi}_1). \tag{B4}$$

Here $\psi=(a_k, d_\alpha)^T$. After the rotation and gauge transformation Eq. (A3) the source and tunneling terms take the form

$$S_t + S_s = \int dt \bar{\psi}_{\gamma}\left[T_{\gamma\delta}(\varphi) + \frac{\kappa}{2}J_{\gamma\delta}(\varphi)\right]\psi_{\delta},$$

$$T_{\gamma\delta} = \begin{pmatrix} \Lambda_c & \Lambda_q \\ \Lambda_q & \Lambda_c \end{pmatrix}, \quad J_{\gamma\delta} = \begin{pmatrix} J_q & J_c \\ J_c & J_q \end{pmatrix}. \tag{B5}$$

Here, indices $c$ and $q$ denote classical and quantum component of a corresponding physical value, i.e., $J_{c,q} = 1/2(J_+ \pm J_-)$ and $\Lambda_\sigma, J_\sigma$ are matrices in a island-lead space

$$\Lambda_\sigma = -\begin{pmatrix} 0 & t_{k\alpha}e^{-i\varphi_\sigma} \\ t_{\alpha k}^{\dagger}e^{i\varphi_\sigma} & 0 \end{pmatrix},$$

$$J_\sigma = \begin{pmatrix} 0 & it_{k\alpha}e^{-i\varphi_\sigma} \\ -it_{\alpha k}^{\dagger}e^{i\varphi_\sigma} & 0 \end{pmatrix}. \tag{B6}$$

It's possible to get rid of highly nonlinear source term Eq. (B3) by a suitable change in field variables. Indeed, one can easily check that up to linear order in $\kappa$

$$T_{11}(\varphi_+, \varphi_-) + \frac{\kappa}{2}J_{11}(\varphi_+, \varphi_-) = T_{11}\left(\varphi_+ + \frac{\kappa}{2}, \varphi_- - \frac{\kappa}{2}\right). \tag{B7}$$

The same property holds for all the elements of matrices $T_{\gamma\delta}, J_{\gamma\delta}$. By making a change

$$\varphi_+ + \frac{\kappa}{2} \to \varphi_+, \quad \varphi_- - \frac{\kappa}{2} \to \varphi_-, \tag{B8}$$

we put the whole $\kappa$ dependence into gaussian part of the action. Then

$$S_s = -\int \kappa(t)\left(\frac{C}{\sqrt{2}}\ddot{\varphi}_c + C_g \dot{U}_c\right)dt. \tag{B9}$$

The average current Eq. (B3) reads

$$\langle I\rangle = \frac{C}{\sqrt{2}}\langle\ddot{\varphi}_c\rangle + C_g \dot{U}_c. \tag{B10}$$

Using Eq. (A5) and to linear order in $U_c(t)$ we find the current to be

$$\langle I\rangle = C_g \dot{U}_c + \frac{C}{\sqrt{2}}\langle\ddot{\varphi}_c\rangle + iCC_g\int U_c(t')dt'\langle\ddot{\varphi}_c(t)\dot{\varphi}_q(t')\rangle. \tag{B11}$$

The admittance becomes

$$\mathcal{G}(\omega) = -i\omega C_g\left[1 + \frac{\Pi_R(\omega)}{C}\right]. \tag{B12}$$

Hence,

$$\text{Im }\Pi_R(\omega) = \frac{C}{C_g}\frac{\text{Re }\mathcal{G}(\omega)}{\omega}. \tag{B13}$$

In the case of spin variables (strong coupling) we can easily get the analogue of formula (B12) for the admittance using the same steps. This way we establish the relation between admittance and spin-polarization operator $\Pi_s$ quoted in the main body

$$\mathcal{G}(\omega) = -i\omega\frac{C_g}{C}\Pi_s^R(\omega), \tag{B14}$$

where $\Pi_s^R(\omega)$ is given by Eq. (A11).

## APPENDIX C: INSTANTON CONTRIBUTIONS

### 1. Massive fluctuations

We expand the fluctuating field $\delta\varphi(\tau)$ in the basis of eigenfunctions $\delta\varphi(\tau)=\Sigma_m C_m \varphi_m(\tau)$, where the basis reads $(u=e^{2\pi iT\tau})$ (Ref. 25)

$$\varphi_m(\tau,z) = u^{m-1}\frac{u-z}{1-u\bar{z}}, \quad m \geq 2,$$

$$\varphi_{-m}(\tau,z) = \frac{1}{u^{m-1}}\frac{1-u\bar{z}}{u-z}, \quad m \geq 2;$$

$$\varphi_1(\tau,z) = \sqrt{1-|z|^2}\frac{1}{u-z},$$

$$\varphi_{-1}(\tau,z) = \sqrt{1-|z|^2}\frac{u}{1-u\bar{z}}. \tag{C1}$$

Here, $\varphi_{\pm 1}(\tau,z)$ are field zero modes. Then the correlator reads

$$\langle T_\tau \delta\dot{\varphi}(\tau)\delta\dot{\varphi}(\tau')\rangle = T\sum_m \int \mathcal{D}z \dot{\varphi}_{-m}(\tau,z)\dot{\varphi}_m(\tau',z) \times \langle C_{-m}C_m\rangle\frac{\mathcal{D}_1}{\mathcal{D}_0}e^{-g/2+2\pi iqW},$$

$$\langle C_{-m}C_m\rangle = \frac{2\pi}{g\omega_{m-1}} = \frac{1}{g(m-1)T}, \quad m > 0,$$

$$\mathcal{D}z = \frac{d^2 z}{1-|z|^2}, \quad |z| \le 1 - \frac{T}{E_c}.$$

Here, $\mathcal{D}_1/\mathcal{D}_0$ is the ratio of fluctuation determinants. Some care should be taken when regularizing them. We used the scheme proposed in Ref. 18

$$\frac{\mathcal{D}_1}{\mathcal{D}_0} = \frac{g^2 E_c}{2\pi^3 T}. \tag{C2}$$

After simple algebra we obtain

$$\frac{1}{2gE_cT} e^{g/2-2\pi i q W} \langle \mathcal{T}_\tau \delta\dot{\varphi}(\tau)\delta\dot{\varphi}(\tau')\rangle_W = \underbrace{\frac{s}{(1-s)^2}\ln\frac{E_c}{T}}_{\mathrm{I}} - \underbrace{\frac{1}{s}\ln^2(1-s)}_{\mathrm{II}} - \underbrace{\frac{2s}{1-s}}_{\mathrm{III}} - \underbrace{\frac{2\ln(1-s)}{1-s}}_{\mathrm{IV}} + \left(s \to \frac{1}{s}\right),$$

$$s = \frac{u}{u'} = e^{2\pi i T(\tau-\tau')}.$$

Expanding this expression into Taylor series over $s$ we get

$$\mathrm{I} = \sum_{n=1}^{\infty} n s^n \ln\frac{E_c}{T},$$

$$\mathrm{II} = 2\sum_{n=1}^{\infty} \frac{s^n}{1+n}\sum_{k=1}^{n}\frac{1}{k} = 2\sum_{n=1}^{\infty}\frac{H_n}{1+n}s^n,$$

$$\mathrm{III} = 2\sum_{n=1}^{\infty} s^n,$$

$$\mathrm{IV} = -2\sum_{n=1}^{\infty} s^n \sum_{k=1}^{n}\frac{1}{k} = -2\sum_{n=1}^{\infty} H_n s^n.$$

Here, $H_n$ is harmonic number. The contribution of Gaussian fluctuations into the correlator becomes

$$\frac{1}{2gE_cT} e^{g/2-2\pi i q W} \langle \mathcal{T}_\tau \delta\dot{\varphi}(\tau)\delta\dot{\varphi}(\tau')\rangle_W = \sum_{n=1}^{\infty} n\left(\ln\frac{E_c}{T} - \frac{2H_n}{1+n}\right)s^n - 2\sum_{n=1}^{\infty} s^n + \left(s \to \frac{1}{s}\right). \tag{C3}$$

Now we make analytical continuation of Fourier components into the region $n \ll 1$. We are interested in linear in $n$ term,

$$H_n = \frac{\pi^2 n}{6} + \mathcal{O}(n^2).$$

Extracting linear part and summing instanton and anti-instanton terms we obtain

$$\langle \mathcal{T}_\tau \delta\dot{\varphi}(\tau)\delta\dot{\varphi}(\tau')\rangle_n = -8gE_c e^{-g/2}\left(1 - \frac{|\omega_n|}{4\pi T}\ln\frac{E_c}{T}\right)\cos 2\pi q + \mathcal{O}(\omega_n^2), \tag{C4}$$

which does cancel partition function renormalization Eq. (41).

### 2. Zero modes

The corresponding single instanton configuration reads

$$\dot{\varphi}_W = 2\pi T W\left(\frac{u}{u-z} + \frac{\bar{z}u}{1-\bar{z}u}\right), \quad W = \pm 1. \tag{C5}$$

The correlator is given by

$$\langle \mathcal{T}_\tau \dot{\varphi}(\tau)\dot{\varphi}(\tau')\rangle_W = e^{-g/2+2\pi i q W}(2\pi T)^2 \frac{\mathcal{D}_1}{\mathcal{D}_0}\int \frac{d^2 z}{1-|z|^2} \times \sum_n \{|z^2|^n s^n + |z^2|^n s^{-n}\}. \tag{C6}$$

The corresponding Fourier component is as follows:

$$\langle \mathcal{T}_\tau \dot{\varphi}(\tau)\dot{\varphi}(\tau')\rangle_n = e^{-g/2} 8\pi^2 T \cos 2\pi q \frac{\mathcal{D}_1}{\mathcal{D}_0}\int \frac{|z^2|^{|n|}}{1-|z|^2} d^2 z. \tag{C7}$$

Expanding it in $n \ll 1$ to linear order we reproduce Eq. (44).

## APPENDIX D: COMPUTATION OF POLARIZATION OPEARTOR

### 1. Lowest order

First we notice that $\mathrm{I}(-\omega_n) = \mathrm{II}(\omega_n)$. Thus we will drop any odd function of $\omega_n$ while calculating $\mathrm{I}(\omega_n)$. The analytical expression for diagram I (see Fig. 5) reads

$$\mathrm{I}(\omega_n) = \frac{gT^2}{4\pi}\sum_{k,m} \frac{|\Omega_m|}{[i(\varepsilon_k + i\omega_n) - \xi_\sigma]^2} \times \frac{1}{i(\varepsilon_k + \omega_n + \Omega_m) - \xi_{-\sigma}}\frac{1}{i\varepsilon_k - \xi_\sigma}.$$

Performing the sum over fermion frequencies we get

$$\mathrm{I}(\omega_n) = \frac{gT}{4\pi}\sum_m |\Omega_m| \left\{ \frac{n_f(\xi_\sigma)}{(i\omega_n)^2}\left[\frac{1}{\Delta\sigma + i(\omega_n + \Omega_m)} - \frac{1}{\Delta\sigma + \Omega_m}\right] - \frac{n_f(\xi_{-\sigma})}{\Delta\sigma + i(\omega_n + \Omega_m)}\frac{1}{(\Delta\sigma + i\Omega_m)^2}\right\},$$

where $n_f(x) = 1/(e^{\beta x}+1)$ is Fermi distribution function. Simple algebra shows that $\mathrm{I}(\omega_n) + \mathrm{II}(\omega_n) = -\mathrm{III}(\omega_n)$. Thus, taking the limit $\eta \to -\infty$ we obtain

$$\mathrm{I}(\omega_n) + \mathrm{II}(\omega_n) + \mathrm{III}(\omega_n) = -\frac{2gT}{\pi}e^{\beta\eta}\sinh\frac{\Delta}{2T}\sum_m |\Omega_m| \times \left\{\frac{1}{(\Delta + i\Omega_m + i\omega_n)(\Delta + i\Omega_m)^2} + \omega_n \to -\omega_n\right\}.$$

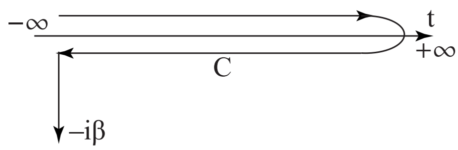


FIG. 10. Keldysh contour.

Now it is clear that the sum over $\Omega_m$ can be taken in terms of digamma functions. The answer is given by Eq. (70).

### 2. Callan-Symanzik equation for $\langle s_z \rangle$

The anomalous dimension $\gamma$ of operator $s^z_{\rm pf}$ is introduced as

$$Z^\gamma s^{z,r}_{\rm pf}(\bar{\Delta},\bar{g}) = s^z_{\rm pf}(\Delta,g,\Lambda), \tag{D1}$$

where $Z$ is given by Eq. (79) and $\Lambda$ is a cutoff, $\Lambda \sim E_c$. To extract $\gamma$ we write down the corresponding CS equation for the Green's function: $F_{\rm pf}(\Delta,g,\Lambda)=\frac{1}{2}\Sigma_\sigma \sigma\langle\bar{\psi}_\sigma\psi_\sigma\rangle$. The tree-level $F_{\rm pf}$ reads

$$F_{\rm pf}(\Delta) = -e^{\beta\eta}\sinh\frac{\Delta}{2T}. \tag{D2}$$

Following general renormalization group philosophy and with the help of Eq. (D1) we write the corresponding CS equation for function $F(\Delta,g,\Lambda)$ in the form

$$\left(\frac{\partial}{\partial \ln\Lambda} + \beta_g\frac{\partial}{\partial g} + \beta_\Delta\frac{\partial}{\partial\Delta} - \gamma\frac{d\ln Z}{d\ln\Lambda}\right)F_{\rm pf}(g,\Delta,\Lambda) = 0, \tag{D3}$$

where the corresponding $\beta$ functions are given by

$$\beta_g = \frac{g^2}{2\pi^2}, \quad \beta_\Delta = \frac{g\Delta}{2\pi^2}. \tag{D4}$$

The term with $\beta_g$ always contains extra $g$ and can be dropped in the leading order. Using action [Eq. (66)] we work out the last term

$$\frac{d\ln Z}{d\ln\Lambda} = -\frac{g}{4\pi^2}. \tag{D5}$$

To find $\gamma$ we need to get $F$ in the next to Eq. (D2) order

$$F_{\rm pf}(\Delta,g,\Lambda) = -e^{\beta\eta}\sinh\frac{\Delta}{2T}\left(1 - \frac{g}{2\pi^2}\ln\frac{\Lambda}{\varepsilon}\right) + e^{\beta\eta}\frac{g\Delta}{4\pi^2 T}\cosh\frac{\Delta}{2T}\ln\frac{\Lambda}{\varepsilon}. \tag{D6}$$

Here, $\varepsilon$ is a characteristic scale of interaction. Plugging Eqs. (D5) and (D6) into Eq. (D3) we find

$$\gamma = 2. \tag{D7}$$

### 3. Exact expression for polarization operator

In order to work out the polarization-operator diagram in Fig. 6 we follow the scheme proposed by Eliashberg.[44] First

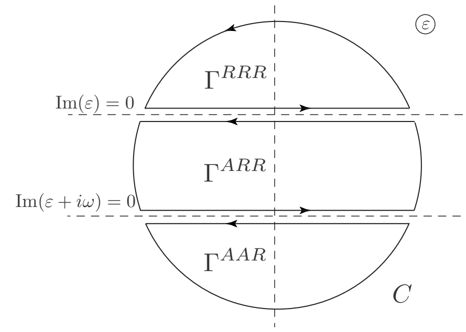


FIG. 11. Contour for polarization operator $\Pi(\omega)$.

we establish the analytical properties of vertex function $\boldsymbol{\Gamma}(z,z+i\omega_n,i\omega_n)$ as a function of complex variable $z$. The operator expression for the vertex function reads

$$\boldsymbol{\Gamma}_\sigma(\tau_1-\tau,\tau_2-\tau) = \langle \mathcal{T}_\tau\bar{\psi}_\sigma(\tau)\psi_\sigma(\tau)\bar{\psi}_\sigma(\tau_1)\psi_\sigma(\tau_2)\rangle. \tag{D8}$$

Its Lehman representation is as follows:

$$\boldsymbol{\Gamma}_\sigma(z,z+i\omega,i\omega) = T^4\sum_{nklm} W^\sigma_{lnmk}W^{*\sigma}_{lkmn}\Bigg[\frac{e^{\beta\omega_{kn}}}{\omega_{kn}-i\omega}\times\left\{\frac{e^{-\beta\omega_k}+e^{-\beta\omega_l}}{z+\omega_{kl}} - \frac{e^{-\beta\omega_l}+e^{-\beta\omega_n}}{z+i\omega-\omega_{ln}}\right\} + \frac{e^{\beta\omega_{lm}}}{\omega_{lm}-i\omega}\left\{\frac{e^{-\beta\omega_n}+e^{-\beta\omega_m}}{z+\omega_{nm}} - \frac{e^{-\beta\omega_l}+e^{-\beta\omega_n}}{z+i\omega-\omega_{ln}}\right\}\Bigg],$$

$$W^\sigma_{lnmk} = \langle l|\psi_\sigma|n\rangle\langle m|\psi_\sigma|k\rangle. \tag{D9}$$

Complex calculus teaches us that the sum Eq. (D9) defines a function with two horizontal cuts: $\mathrm{Im}(z+i\omega)=0$ and $\mathrm{Im}(z)=0$. For simplicity let us restrict our attention to a *retarded* vertex function $\omega_n>0$. Next we define three vertex functions in accordance with the structure of cuts

$$\boldsymbol{\Gamma}^{\rm RRR}(z,z+i\omega,i\omega) \quad \text{if} \quad \mathrm{Im}\, z>0,$$

$$\boldsymbol{\Gamma}^{\rm ARR}(z,z+i\omega,i\omega) \quad \text{if} \quad -i\omega_n<\mathrm{Im}\, z<0,$$

$$\boldsymbol{\Gamma}^{\rm AAR}(z,z+i\omega,i\omega) \quad \text{if} \quad \mathrm{Im}\, z<-i\omega_n. \tag{D10}$$

The general expression for $\Pi_\sigma(i\omega_n)$ then becomes

$$\Pi_\sigma(i\omega_n) = \frac{T}{4}\sum_{\varepsilon_k}\boldsymbol{\Gamma}_\sigma(i\varepsilon_k,i\varepsilon_k+i\omega_n,i\omega_n)G_\sigma(i\varepsilon_k+i\omega_n)\times G_\sigma(i\varepsilon_k) = \oint_C\frac{d\varepsilon}{16\pi i}\tanh\frac{\varepsilon}{2T}\boldsymbol{\Gamma}_\sigma\times(\varepsilon,\varepsilon+i\omega_n,i\omega_n)G_\sigma(\varepsilon+i\omega_n)G_\sigma(\varepsilon). \tag{D11}$$

The contour $C$ is shown in Fig. 11. As usual the integral over large circle vanishes and we are left with integrals over different branches

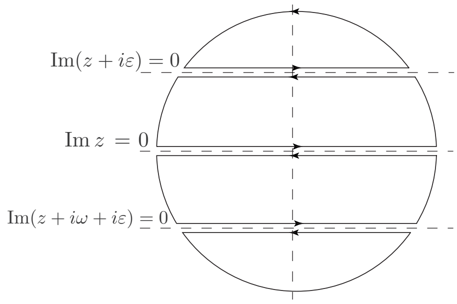


FIG. 12. Contour for the vertex function $\Gamma^{\mathrm{ARR}}$.

$$\Pi_\sigma(i\omega_n) = \frac{1}{4}\int_{-\infty}^{\infty}\frac{d\varepsilon}{4\pi i}\tanh\frac{\varepsilon}{2T}\{\mathbf{\Gamma}_\sigma^{\mathrm{RRR}}(\varepsilon,\varepsilon+i\omega_n,i\omega_n) \times G_\sigma^R(\varepsilon+i\omega_n)G_\sigma^R(\varepsilon) - \mathbf{\Gamma}_\sigma^{\mathrm{ARR}}(\varepsilon,\varepsilon+i\omega_n,i\omega_n) \times G_\sigma^R(\varepsilon+i\omega_n)G_\sigma^A(\varepsilon) + \mathbf{\Gamma}_\sigma^{\mathrm{ARR}}(\varepsilon-i\omega,\varepsilon,i\omega_n) \times G_\sigma^R(\varepsilon)G_\sigma^A(\varepsilon-i\omega_n) - \mathbf{\Gamma}_\sigma^{\mathrm{AAR}}(\varepsilon-i\omega_n,\varepsilon,i\omega_n) \times G_\sigma^A(\varepsilon)G_\sigma^A(\varepsilon-i\omega_n)\}. \quad \text{(D12)}$$

Making analytical continuation $i\omega_n\to\omega+i0$ we get result (85).

### 4. Dyson equation for the vertex

Following the same scheme as in the previous section we derive the expression for the vertex function. The contour $C$ depends on the type of the vertex we need to get from Eq. (87). The contour for the vertex $\mathbf{\Gamma}_\sigma^{\mathrm{ARR}}$ is depicted in Fig. 12.

The result reads

$$\mathbf{\Gamma}_\sigma^{\mathrm{ARR}}(\varepsilon,\varepsilon+\omega,\omega) = 1 - \mathrm{I} + \mathrm{II} - \mathrm{III}, \quad \text{(D13)}$$

where

$$\mathrm{I} = \int_{-\infty}^{\infty}\frac{dx}{4\pi i}G_{-\sigma}^A(x)G_{-\sigma}^R(x+\omega)\mathbf{\Gamma}_{-\sigma}^{\mathrm{ARR}}(x,x+\omega,\omega) \times\left[2i\,\mathrm{Im}\,\alpha_R(x-\varepsilon)\coth\frac{x-\varepsilon}{2T} - \tanh\frac{x}{2T}\alpha_R(x-\varepsilon) + \tanh\frac{x+\omega}{2T}\alpha_A(x-\varepsilon)\right],$$

$$\mathrm{II} = \int_{-\infty}^{\infty}\frac{dx}{4\pi i}\alpha_R(x-\varepsilon)G_{-\sigma}^R(x)G_{-\sigma}^R(x+\omega) \times\mathbf{\Gamma}_{-\sigma}^{\mathrm{RRR}}(x,x+\omega,\omega)\tanh\frac{x}{2T},$$

$$\mathrm{III} = \int_{-\infty}^{\infty}\frac{dx}{4\pi i}\alpha_A(x-\varepsilon)G_{-\sigma}^A(x)G_{-\sigma}^A(x+\omega) \times\mathbf{\Gamma}_{-\sigma}^{\mathrm{AAR}}(x,x+\omega,\omega)\tanh\frac{x+\omega}{2T}. \quad \text{(D14)}$$

Here, function $\alpha(z)$ is an interaction propagator whose Matsubara counterpart is shown in Fig. 4. As usual it has a cut $\mathrm{Im}\, z=0$ which allows to define two functions

$$\alpha_R(\omega) = \bar{g}\frac{i\omega}{4\pi}, \quad \alpha_A(\omega) = -\bar{g}\frac{i\omega}{4\pi}. \quad \text{(D15)}$$

The integrands entering terms II and III are explicitly analytical in the upper and lower halves of the complex plane, respectively. Consequently we may turn the corresponding integrals in to sums over Matsubara frequencies $i\varepsilon_n$. Next one can easily prove the following identities:

$$\mathbf{\Gamma}_\sigma^{\mathrm{RRR}}(i\varepsilon_n,i\varepsilon_n+\omega,\omega) = \mathbf{\Gamma}_\sigma^{\mathrm{ARR}}(i\varepsilon_n,i\varepsilon_n+\omega,\omega),$$

$$\mathbf{\Gamma}_\sigma^{\mathrm{AAR}}(i\varepsilon_n-\omega,i\varepsilon_n,\omega) = \mathbf{\Gamma}_\sigma^{\mathrm{ARR}}(i\varepsilon_n-\omega,i\varepsilon_n,\omega). \quad \text{(D16)}$$

This way we drastically simplify our Dyson equation by rewriting it entirely in terms of a single vertex $\mathbf{\Gamma}_\sigma^{\mathrm{ARR}}$. Next,

$$\mathrm{II} = \frac{\bar{g}i}{4\pi}\sum_{\varepsilon_n}\mathbf{\Gamma}_{-\sigma}^{\mathrm{ARR}}(i\varepsilon_n,i\varepsilon_n+\omega,\omega)\frac{i\varepsilon_n-\varepsilon}{(i\varepsilon_n-\xi_{-\sigma})(i\varepsilon_n+\omega-\xi_{-\sigma})},$$

$$\mathrm{III} = \frac{\bar{g}i}{4\pi}\sum_{\varepsilon_n}\mathbf{\Gamma}_{-\sigma}^{\mathrm{ARR}}(i\varepsilon_n-\omega,i\varepsilon_n,\omega) \times\frac{i\varepsilon_n-\omega-\varepsilon}{(i\varepsilon_n-\xi_{-\sigma})(i\varepsilon_n-\omega-\xi_{-\sigma})}. \quad \text{(D17)}$$

As usual regularization scheme allows us to drop these sums. The integrand of term I however contains $G^AG^R$. As a consequence it is singular at $\omega,g\to 0$ as explained in the main body. This way we recover Eq. (88).

## APPENDIX E: RATE PROBABILITIES

To work out rates $\Gamma^0$ and $\gamma$ we follow standard scheme. We introduce Heisenberg $\psi$ operators according to

$$\psi_d(t) = \sum_\alpha d_\alpha e^{-i\varepsilon_\alpha t}, \quad \psi_a(t) = \sum_k a_k e^{-i\varepsilon_k t}. \quad \text{(E1)}$$

Then the matrix elements in the basis of filling numbers become

$$\langle 0|\psi_d|1\rangle = \sum_\alpha \langle 0|d_\alpha|1\rangle e^{-i(\varepsilon_\alpha+\Delta)t}. \quad \text{(E2)}$$

In an ordinary fashion we change the Hamiltonian by gauge transformation of fermion fields (path-integral approach is implied),

$$\psi_d(t) \to \psi_d(t)e^{iC_g/C\int U(t)dt}, \quad \text{(E3)}$$

where $U(t)=U_\omega\cos\omega t$. Now the whole $U(t)$ dependence is transferred into the tunneling part of the Hamiltonian

$$H_t = \sum_{k,\alpha} t_{k\alpha}a_k^\dagger d_\alpha e^{iC_g/C\int U(t)dt} + \mathrm{H.c.} \quad \text{(E4)}$$

Let us compute rate $\Gamma_{10}(t)$. The initial and final states read

$$|i\rangle = |k,N\rangle,$$

$$|f\rangle = d_\alpha^\dagger a_k|k,N\rangle. \tag{E5}$$

Here, $k(N)$ is the number of electrons in the island (lead). As usual, $S$-matrix formalism gives the necessary amplitude in the form

$$A_{10}(t) = -i\langle f|\int_{-\infty}^{t} H_t(t)dt|i\rangle = -i\langle i|a_k^\dagger d_\alpha\int_{-\infty}^{t} H_t(t)dt|i\rangle. \tag{E6}$$

Now we substitute $\int U(t)dt=(U_\omega/\omega)\sin\omega t$ and tunneling Hamiltonian assumes the form

$$H_t = \sum_{k,\alpha} t_{k\alpha}a_k^\dagger d_\alpha\left(1+\frac{iC_gU_\omega}{C\omega}\sin\omega t\right)+\text{H.c.} \tag{E7}$$

The detailed-balance relations for probability rates read

$$\Gamma_{01}^0(\Delta)=\Gamma_{10}^0(-\Delta),$$

$$\gamma_{01}(t,\Delta) = -\gamma_{10}(t,-\Delta). \tag{E8}$$

Plugging Eq. (E7) into Eq. (E6) and integrating one gets the amplitude of transition $0\to1$

$$A_{10}(t) = t_{\alpha k}^\dagger(1-n_\alpha)n_k e^{i(\varepsilon_\alpha-\varepsilon_k+\Delta)t}\left\{\frac{1}{\varepsilon_k-\varepsilon_\alpha-\Delta+i0}-\frac{C_gU_\omega}{2C\omega}\times\left[\frac{e^{i\omega t}}{\varepsilon_k-\varepsilon_{k'}-\Delta-\omega+i0}-\frac{e^{-i\omega t}}{\varepsilon_k-\varepsilon_{k'}-\Delta+\omega+i0}\right]\right\}. \tag{E9}$$

Squaring it, taking thermal average and integrating we get the full expression for the probability in the linear-response regime

$$W_{10}(t) = \frac{g}{8\pi^2}\int\frac{sds}{e^{\beta s}-1}\left\{\frac{e^{2\lambda t}}{(s-\Delta)^2+\lambda^2}-\frac{C_gU_\omega}{C\omega}\times\frac{1}{s-\Delta+i0}\left(\frac{e^{-i\omega t}}{s-\Delta-\omega-i0}-\frac{e^{i\omega t}}{s-\Delta+\omega-i0}\right)\right\}+\text{c.c.}, \tag{E10}$$

where $g$ is defined in Eq. (12). Now we find the transition rate as a derivative of a transition probability $\Gamma_{10}(t)=dW_{10}(t)/dt$ and the following expression for $\gamma s$:

$$\gamma_{10}(\omega) = -\frac{g}{2\pi}\int_{-\infty}^{\infty}\frac{ds}{2\pi i}\frac{s}{e^{\beta s}-1}\times\left[\frac{1}{s-\Delta+i0}\frac{1}{s-\Delta-\omega-i0}-\frac{1}{s-\Delta-i0}\frac{1}{s-\Delta+\omega+i0}\right]. \tag{E11}$$

The integrand converges very well in the complex plane and the integral can be easily taken,

$$\gamma_{10}(\omega) = -\frac{g}{2\pi}\left[\frac{1}{\omega+i0}\left(\frac{\Delta+\omega}{e^{\beta(\Delta+\omega)}-1}-\frac{\Delta}{e^{\beta\Delta}-1}\right)+T\sum_{n=1}^{\infty}\frac{i\omega_n}{i\omega_n-\Delta}\left(\frac{1}{i\omega_n-\Delta-\omega}-\frac{1}{i\omega_n-\Delta+\omega}\right)\right]. \tag{E12}$$

Expressing the sum in terms of digamma functions we get Eq. (106). With the help of Eqs. (E12) and (E8) one can establish the following useful identity:

$$\gamma_{10}(\omega)-\gamma_{01}(\omega)=\frac{g}{2\pi}. \tag{E13}$$


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